Mathematics workshop 3

Session 1: Mental mathematics

Objectives

By the end of this session you will:

- have considered what is meant by mental mathematics
- have considered some strategies for helping students to improve their skills in doing mental mathematics

1.1

What is mental mathematics?

- Memorising and recalling facts
- Using mental processes to do calculations
- Using visual imagery
- Estimating
- Interpreting
- Recognising patterns
- Conjecturing and hypothesising

1.2

Pedagogical issues

Students need:

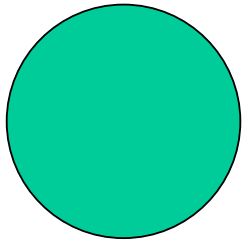
- visual images to aid memorisation
- frequent practice in recall of factual knowledge
- teaching of mental methods and strategies
- practice in explaining their thinking orally and in writing
- opportunities to discuss their methods with peers

1.3

Explicit information

Look at this circle.

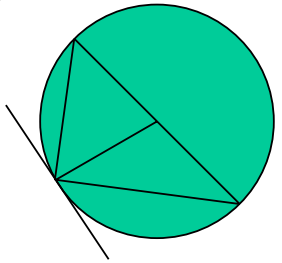
What can you see?



1.4

Implicit information

I can imagine: centre, radius, diameter, tangent, angle in a semicircle …



1.5

Explicit information

What can you see?

1.6

Implicit information

I can see squares, rectangles, faces, vertices, …

I can imagine a solid, faces, net, …

1.7

Explicit information

What can you see?

$x^2 - y^2$

$ax^2 + bx + c = 0$

$y = ax^3$

$f(x) = x^2$; what can you say about $f(-x)$?

1.8

Mathematics workshop 3

Session 2: Proportional reasoning

Objectives

By the end of this session you will:

- have considered some strategies for teaching students about proportional reasoning
- have seen the need for a consistent approach
- have seen how visualisation can aid understanding

2.1

Pedagogical issues

Frequent problems encountered by students:

- using addition instead of multiplication
- misinterpreting the meaning of proportion
- having difficulties with fractions
- not being able to rearrange formulae

2.2

Compare

Compare

to

two to three

2 : 3

smaller

2.3

Compare

Compare

to

three to two

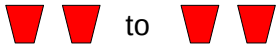
3 : 2

bigger

2.4

Compare

Compare



to

two to two

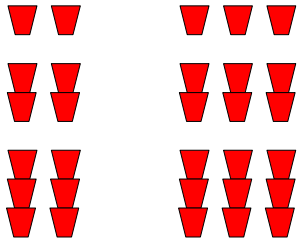
2 : 2

same

2.5

Compare

Compare



2 : 3

2.6

Compare

Compare

x	y
2	3
3	4.5
4	6
5	7.5
6	9
7	10.5
n	$3n/2$

2.7

Compare

Compare

$$y \propto x$$

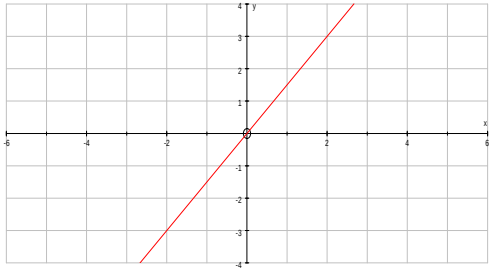
$$y = kx$$

$$y = \frac{3}{2}x$$

2.8

Compare

Compare



2.9

Proportional reasoning

1. Direct proportion

$$y = kx$$

$$y_1 = kx_1$$

$$y_2 = kx_2$$

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}$$

$$y_1 : y_2 = x_1 : x_2$$

2. Indirect proportion

$$y = \frac{k}{x}$$

$$y_1 = \frac{k}{x_1}$$

$$y_2 = \frac{k}{x_2}$$

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}$$

$$y_1 : y_2 = x_2 : x_1$$

2.10

Mathematics workshop 3

Session 3: Geometric reasoning

Objectives

By the end of this session you will:

- have considered some strategies for teaching students how to use implicit knowledge to reason in geometry
- have seen how practical work can help students understand mathematical concepts, rules and formulae

3.1

Pedagogical issues

Students often encounter difficulties in:

- visualising 3-D shapes
- visualising nets to make solids
- memorising formulae
- seeing the relationship between the various solid shapes
- knowing the difference between a prism and a pyramid

3.2

Mathematics workshop 3

Session 4: Geometric deduction

Objectives

By the end of this session you will:

- have considered some strategies for teaching students how to use explicit and implicit information
- have seen how a consistent approach is essential for understanding the concept of deduction

4.1

Pedagogical issues

Frequent problems encountered by students:

- understanding the meaning of *parallel*
- understanding what an angle measures
- knowing the difference between explicit and implicit information
- knowing how to form a mathematical argument

4.2

Mathematics workshop 3

Session 5: Generating graphs

Objectives

By the end of this session you will:

- have considered some strategies for teaching graphs of functions using ICT
- have seen how visualisation can improve students' understanding of graphs

5.1

Pedagogical issues

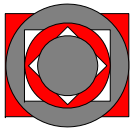
Teachers ask:

- Will using ICT make students lazy?
- Will using ICT stop students from thinking?

5.2

Mathematics workshop 3

Session 6: Working with graphs



Objectives

By the end of this session you will:

- have considered some strategies for teaching students how to solve equations
- have seen how visualisation can help students understand what is meant by a solution

6.1

Pedagogical issues

What do we mean when we ask students to 'find a solution'?

For example:

- Solve $x + 3 = y$
- Solve $x + 3 = 7$
- Solve $x^2 + 3x - 7 = y$
- Solve $x^2 + 3x - 7 = 0$
- Solve $x^2 + 3x - 7 = 10$

6.2

Mathematics workshop 3

Session 7: Trigonometry

Objectives

By the end of this session you will:

- have considered some strategies for teaching trigonometry
- have seen how doing practical work and using ICT can provide visual imagery that aids memorisation

7.1

Pedagogical issues

Common difficulties in learning trigonometry:

- the concept of ratio
- rearranging formulae
- orientation of triangles
- memorisation
- signs of angles over 90°

7.2

Mathematics workshop 3

Session 8b: Fractions, decimals and percentages

Objectives

By the end of this session you will:

- have discussed how students learn
- have seen how visualisation is a key component in teaching fractions
- have considered some strategies for teaching that are consistent across grades

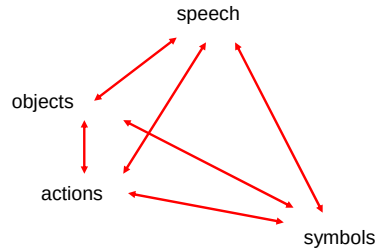
8b.1

A model of learning

Capacity begins	Capacity developing	Capacity developed	Capacity declines
Help given by more capable others, e.g. parents teachers peers coaches experts	Help provided by self	Internalised Automatic Fossilised	Return to earlier stages

8b.2

A model of learning maths



8b.3

Mathematics workshop 3

Session 9b: Calculators

Objectives

By the end of the session you will:

- have considered how to teach students to use a calculator, and some of the difficulties that they can have
- have tried out some calculator activities to use in lessons

9b.1

Knowing which operation to use

- I have to travel 254 km. I stop after 103 km.
  How much further do I have to go?
- Calculate the mean height of 8 people if the sum of their heights is 1248 cm.
- The area of a triangle is 414 cm^2. The height is 9 cm.
  What is its base?
- 12% of my weight is 8.4 kg.
  What is my weight?
- The area of a circle is 225 cm^2.
  What is its radius to the nearest centimetre?

9b.2

Having a sense of the answer

- 3.142 – 0.02
- 709 283 + 815
- 85 × 29 × 11
- 598 ÷ 28
- 485 × 310 ÷ 17

9b.3

Arithmetical laws and conventions

- 2 + 3 × 4
  20 or 14? Why?
- Use a single multiplication to calculate:

  45 × 19 + 45 × 27

  72.5 × 15.3 – 72.5 × 18.7

  38.9 × 17 – 79 × 1.7

9b.4

Knowing when to use a calculator

- 76.2 × 1000
- 404 ÷ 0.01
- 9 × 8.4 – 9 × 3.4
- 89 × 5
- 4.6 × 5
- 85 × 99
- 42 × 25

9b.5

Making connections

- Place value
- +, −, ×, ÷
- Inverse operations
- Fractions, decimals and percentages
- Estimation and approximation
- Substituting into formulae
- Calculating measurements

9b.6

Mathematics workshop 3

Session 8c: Proof

Objectives

By the end of this session you will have:

- considered the significance of proof
- recognised features of different types of proof
- generated some proofs
- considered pedagogical issues in teaching proof

8c.1

Mathematical proof

- What is proof?
- Is demonstration proof?
- Why is proof important?

8c.2

What is proof?

- A proof is a convincing argument based on logical reasoning. It establishes the truth of a mathematical assertion:
  - from already accepted mathematical assertions

    or
  - from some basic assertions (axioms) whose validity is assumed to be self-evident
- A demonstration is not a proof. It makes a result plausible without the necessary reasoning to establish proof

8c.3

Why is proof important?

- Mathematics must be self-consistent; the validity of one statement is used to establish the validity of another
- Mathematics consists of a hierarchy of valid and self-consistent statements
- Mathematics is a valid tool for modelling the physical world only if the model is precisely specified and unambiguous, and gives well-defined answers to problems
- The self-consistency of the model depends on the validity of the mathematics
- Validity = proof

8c.4

Pedagogical issues 1

- Justification, and for whom?
- Demonstration is not proof
- Motivation
- Proving in every lesson
- Seeing the way through
- More ways than one

8c.5

Pedagogical issues 2

- Proof in algebra and in geometry: same or different?
- Vocabulary
- Mathematical notation and reasoning
- Applications of mathematics
- Assessment

8c.6

Vocabulary of proof 1

- Theorem
- Axiom
- Postulate
- Corollary
- Lemma

8c.7

Vocabulary of proof 2

- Logical implication
- Negation
- Therefore
- If and only if
- Necessary and sufficient
- Proof
- Demonstration

8c.8

Common forms of proof

- Logical deduction
- Contradiction
- Induction
- Exhaustion
- Use of counter-example

8c.9

The most famous theorem

- What do you think is the most famous theorem in school mathematics?

8c.10

The Pythagorean theorem

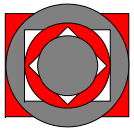
- What is it?
- What does it tell us about geometry?
- In how many ways can you prove it?
- An interesting website:

  www.mathsnet.net/dynamic/Pythagoras

8c.11

Mathematics workshop 3

Session 9c: Understanding and using functions



Objectives

By the end of this session you will have:

- considered the significance of functions and their applications
- recognised and described key features of different functions
- generated some functions from situations and from other functions
- considered some of the pedagogical issues in teaching about functions

9c.1

Functions

- What is a function in mathematics?
- Describe different ways of representing functions
- Suggest some families of functions
- Suggest some ways of categorising functions
- What mathematical features of functions are often important?

9c.2

Generating functions

- By forming composite functions
- By superposition (or other combination) of functions
- From piecewise definitions of the domain
- From physical situations

9c.3

Pedagogical issues 1

- Need for precision in describing functions and solving related equations
- Importance of the three-part nature of a function: the domain, rule and range
- Importance of using correct and relevant vocabulary
- Common features of members of families of functions
- Mathematical notations for functions
- Mathematical analysis of functions
- Alternative representations of functions

9c.4

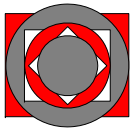
Pedagogical issues 2

- Making connections and comparisons
- Functions as mathematical models of physical contexts
- Further analysis and interpretation of functions in physical contexts
- Differentiation and extension activities
- The role of ICT in the teaching and learning of functions
- Assessment of understanding of the theory and applications of functions

9c.5

Mathematics workshop 3

Session 10c: Applications of calculus



Objectives

By the end of this session you will have:

- considered the significance of analytical calculus in mathematics and its applications
- used calculus to describe features of polynomial and other functions
- solved simple optimisation problems
- looked at some mathematical models

10c.1

Origins

- Why was calculus invented?
- The derivative
- Continuity and differentiability
- Indefinite and definite integrals
- The fundamental theorem of calculus

10c.2

Polynomial functions

- $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$
- What can calculus tell us about *n*th order polynomials?

10c.3

Other functions

- $y = |x|$
  where $|x|$ is the absolute value of x
- $y = x|x|$
- $y = [x]$
  where $[x]$ is the greatest integer less than or equal to x

10c.4

Optimisation

- Think first!
- Know which area, volume, ... to maximise/minimise
- Sketches are helpful
- Identify the variables
- Set up the formula for the function to optimise
- Eliminate unwanted variables by substitution
- Find conditions for local maxima/minima
- Check function values against domain values that have been identified

10c.5

Mathematical models

- $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g'(x))$
- $\int f'(g'(x))\,dx = f(g(x)) + c$

10c.6

Pedagogical issues 1

- Justification
- Motivation
- Seeing the way through
- More ways than one

10c.7

Pedagogical issues 2

- Vocabulary
- Mathematical notation and reasoning
- Applications of mathematics
- Assessment

10c.8

Mathematics workshop 3

Session 11: Collecting data

Objectives

By the end of this session you will:

- have considered strategies for teaching students how to plan a statistical investigation
- have considered data types and how to collect relevant data for a statistical investigation

11.1

Pedagogical issues

Frequent problems encountered by students:

- defining the problem to be investigated
- deciding what information is needed and what variables to use
- deciding how to collect, measure and organise the data
- analysing the data
- using appropriate displays
- making inferences
- making further refinements

11.2

The data handling cycle

- Formulate problem
- Collect data
- Analyse data
- Make inferences from data
- Refine the problem
- Collect more data ...

11.3

Choosing what to investigate

- What projects will motivate students?
- What makes a project worthwhile?
- Who decides what project – the student or the teacher?

11.4

What data?

- Formulate the problem carefully
- Decide on the data required
- Identify the data types
- Decide the methods of data collection
- Decide how to record and organise the data

11.5

Collecting data

- Sample a population
- Carry out a survey involving the design and use of a questionnaire
- Conduct an experiment and take repeated measurements
- Simulate an experiment
- Use secondary sources

11.6

Mathematics workshop 3

Sessions 12 and 13:
Working with data

Objectives

By the end of this session you will:

- have considered strategies for teaching students to use data in statistics
- have discussed how data is analysed, represented and interpreted
- have done some calculations with data

12.1

The data handling cycle

- Formulate problem
- Collect data
- Analyse data
- Make inferences from data
- Refine the problem
- Collect more data ...

12.2

Pedagogical issues 1

Common difficulties met by students:

- defining the problem to be investigated
- planning the investigation
- deciding how to collect, measure and organise the data
- deciding on the most appropriate statistics and diagrams
- analysing the data

12.3

Pedagogical issues 2

Common problems for students:

- comparing data
- deciding whether to use ICT
- doing the calculations
- working to a sensible level of accuracy
- interpreting the calculations and making inferences
- making further refinements

12.4

Useful websites

United Kingdom

- www.stats4schools.gov.uk
- www.censusatschool.ntu.ac.uk
- www.coventry.ac.uk/discuss
- www.statistics.gov.uk

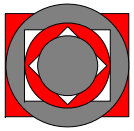
United States

- www.fedstats.gov

12.5

Mathematics workshop 3

Session 14: Probability



Objectives

By the end of this session you will:

- have considered difficulties students experience in learning about probability
- have considered strategies for teaching probability
- have discussed the language of probability
- have looked at random variables and probability distributions
- have done some calculations of probability

14.1

Some difficulties

- Probability scale
- Technical language
- Popular language
- Identify the sample space
- Add or multiply
- Discrete versus continuous
- Which distribution
- Using probability to make inferences

14.2

Random variables and distributions

- Discrete and continuous random variables
- Discrete probability distribution
- Continuous probability distribution
- Probability density function
- Symmetric probability distribution
- Skewed probability distribution

14.3

Discrete probability distribution

- X is a discrete random variable with probability distribution

Value of X	$x_1, x_2, x_3, \ldots, x_k$
Probability	$p_1, p_2, p_3, \ldots, p_k$

- Each p_i is in [0, 1]
- $p_1 + p_2 + p_3 + \ldots + p_k = 1$
- Expectation = $x_1p_1 + x_2p_2 + x_3p_3 + \ldots + x_kp_k$ = mean of X

14.4

Continuous random variable

- Takes all values in an interval of numbers
- The probability distribution is called a density curve
- The total area under the density curve is 1
- If a subset of the number interval denotes event A, then P(A) is the area bounded by:
  - the curve
  - the values of the random variable at the boundaries of the subset
  - the random variable axis

14.5

Some distributions

Continuous distributions

- Uniform
- Normal

Discrete distributions

- Binomial
- Poisson
- Geometric

14.6

Rules of probability 1

- For any event A, $0 \le P(A) \le 1$
- For sample space S, $P(S) = 1$
- $P(\sim A) = 1 - P(A)$

14.7

Rules of probability 2

- Addition law
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Multiplication law
  $P(A \cap B) = P(B \mid A)\,P(A)$
- Bayes' theorem
  $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)\,P(A)}{P(B \mid A)\,P(A) + P(B \mid \sim A)\,P(\sim A)}$$

14.8

Mathematics workshop 3

Session 15: Statistical inference

Objectives

By the end of this session you will:

- have been introduced to the idea of statistical inference
- have considered some difficulties in teaching and learning statistical inference
- have discussed introductory techniques and used the language of statistical inference
- have made some statistical inferences

15.1

Population and sample

Terminology

- A **parameter** is a number that describes the population
- A **statistic** is a number that describes the sample

Difficulties

- Although a parameter is fixed, its value is unknown
- Although a statistic is known, its value may vary from sample to sample

15.2

Sampling

- Avoid bias
- Keep variability down
- Four main possibilities:
  - high bias, low variability
  - high bias, high variability
  - low bias, high variability
  - low bias, low variability

15.3

Sampling distribution of a statistic

- The sampling distribution of a statistic is the distribution of values taken by all possible samples of the same size taken from the same population

15.4

Central limit theorem

- Simple random samples of size n taken from a population with mean μ and variance σ have a sampling distribution of the sample mean that is exactly normal if the population itself is normal and approximately normal when n is large
- The sample mean in either case has the distribution

$$\bar{x} \sim \mathrm{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

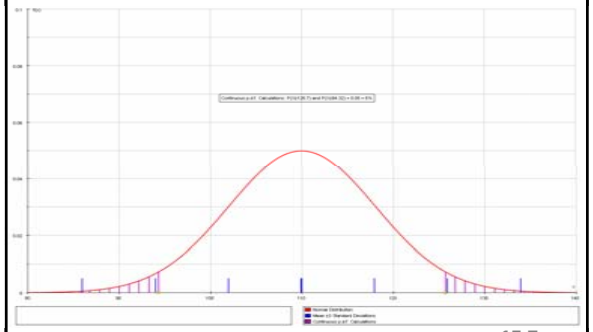
15.5

Confidence interval

- A random sample of 25 students in a large statistics class showed that they spent on average 110 minutes per week studying statistics, with a standard deviation of 40 minutes
- Give a 95% confidence interval for the mean time spent on statistics by students in the class
- Do 95% of the students study for times that lie in this interval?

15.6

95% confidence interval



15.7

95% of the time

Interpretation

- The method (in the long run) gives correct results 95% of the time. So:

  EITHER

  – The true mean lies in the interval [94.32, 125.68]

  OR

  – The simple random sample was one of the 5% of samples for which its mean was not within 15.68 of the true population mean

15.8

Confidence interval for population mean

- Population with known standard deviation σ and unknown mean μ
- Simple random sample of size n and mean x-bar
- A level C confidence interval for μ is

  $$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

- C is the area under the standard normal curve between $-z^*$ and z^*

15.9

Margin of error

- The user chooses the confidence level and so the margin of error
- High confidence level and low margin of error are desirable
- To reduce the margin of error:
  - use lower level of confidence
  - increase sample size
  - reduce the standard deviation

15.10

Problem

- Do middle-aged male teachers have higher blood pressure than the general population of middle-aged males?

  The mean systolic blood pressure of a population of middle-aged men is 130 and the standard deviation is 15.

  A sample of 36 middle-aged male teachers is found to have a mean systolic blood pressure of 132.57.

  Is this evidence for believing that the blood pressure of middle-aged male teachers differs from the mean blood pressure of middle-aged men in the population?

15.11

Hypotheses

Null hypothesis (H_0)

- The mean blood pressure of the population of middle-aged male teachers is 130

Alternative hypothesis (H_1)

- The mean blood pressure of the population of middle-aged male teachers is not equal to (is greater than/is less than) 130

15.12

Test statistic

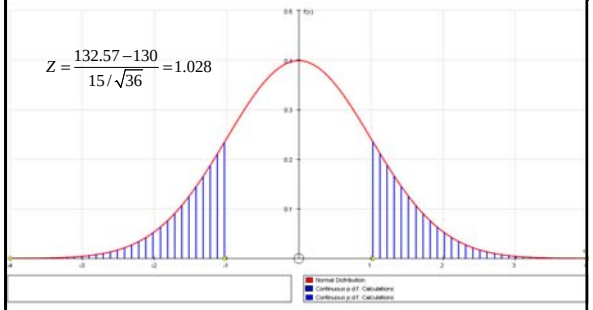
Notation

- Population (of middle-aged male teachers) mean μ
- Hypothesised value of population mean $\mu_0 = 130$
- Population standard deviation $\sigma = 15$
- Sample size $n = 36$
- Sample mean $\bar{x} = 132.57$
- Test statistic (the standardised sample mean) $z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$
- Null hypothesis $H_0 : \mu = \mu_0$
- Alternative hypothesis $H_1 : \mu \neq \mu_0$

15.13

Calculation 1

$$Z = \frac{132.57 - 130}{15/\sqrt{36}} = 1.028$$



15.14

Calculation 2

- P($Z \geq 1.028$) = 1 – 0.8480 = 0.1520
- P($Z \leq -1.028$) = 1 – 0.8480 = 0.1520
- P($Z \leq -1.028$ or $Z \geq 1.028$) = 2 × 0.1520 = 0.3040
- More than 30.4% (this is called the P-value) of the time a random sample of 36 middle-aged men would have a mean blood pressure at least as far from 130 as the sample of middle-aged male teachers
- Accept hypothesis H_0 that the teachers are no different than the general population

15.15

P-value and level of significance

- The P-value is the probability, based on the assumption that the null hypothesis is true, that the test statistic takes a value at least as great as the observed value
- The smaller the P-value, the greater the evidence is against the null hypothesis and in favour of the alternative hypothesis
- If the P-value $\leq \alpha$, the data are said to be statistically significant at level α

15.16

Mathematics workshop 3

Session 16: Place value

Objectives

By the end of this session you will:

- have viewed and discussed some ICT resources
- have considered how the workshop should be followed up in school

16.1

Support for teachers

The long-term task is to help teachers to:

- continue to become familiar with the standards, particularly new subject matter
- refine and develop the scheme of work and lesson plans
- strengthen teaching methodology and assessment
- select and become confident with the use of a wide range of teaching and learning resources, including ICT
- become familiar with the new national tests and how to use results to improve teaching and learning

16.2

Uses of ICT in mathematics

- Basic, scientific, statistical and graphics calculators
- Function graph plotters, e.g. *Autograph*
- Programming languages, e.g. *Logo*
- Dynamic geometry systems, e.g. *Geometer's sketchpad*
- Spreadsheets, databases, e.g. Excel
- Presentation software, e.g. PowerPoint
- Small programs
- The Internet
- Interactive whiteboards

16.3