Mathematics lessons for Grade 11

Lessons in this section

Using these lesson plans

Each of these sample lessons for Grade 11 is suitable for use with a whole class. The lessons are standalone examples to illustrate different teaching and learning activities. They are not intended to be taught as a sequence.

The first two lessons are based on the Grade 11 standards for foundation mathematics. The third and fourth lessons are based on the Grade 11 standards for advanced mathematics. The relevant standards are shown in the lesson plans.

The lessons are drawn from different topics to show spread rather than sequence. They illustrate different pedagogical styles for the various topics and the different sets of standards.

Each lesson has been organised in three parts: a starter to introduce the lesson, a main activity, and a final phase to help students to reflect on the lesson and consolidate their learning. Early in the lesson, you should outline the purpose of the lesson, drawing out for students what they will learn and how this builds on previous work. In the final part of the lesson, you will need to establish the key learning points, what students should remember and what they will go on to learn next about the topic. You should also help students to appreciate the links that can be made to other topics in mathematics, in other subjects or in the real world.

The lesson plan suggests how you as the teacher might teach the class and interact with students but you will need to add your own questions to present and develop the mathematical threads that run through the lesson. The plan makes no attempt to tell students how or when to take notes of the lesson.

The lesson plans do not include homework tasks because the lessons are single examples taken out of sequence. You will need to provide homework since it is an important part of a lesson.

There may be too much material in the lesson plan for one lesson (e.g. of 60 minutes), since the rate of teaching and learning will depend on the class. In this case, you might designate one of the activities in the lesson as homework, or carry it forward to the next lesson. Be selective about which activity to cut – it does not have to be the last one merely because it comes at the end.

You may need to supplement the activities in the lesson with simpler or more challenging tasks if students in your class have a range of attainment. You could choose from activities in textbooks or from your own resources. If you wish, different tasks can be given to different groups of students, according to their needs. Each lesson plan includes some suggestions for further tasks.

Answers to some questions are provided to help you to correct students' responses and give feedback. Many of the questions are open-ended and have more than one possible answer. They are there to facilitate critical discussion, to help students to generalise and to help them gain insight. They illustrate style and a pedagogical approach rather than the words you should use yourself.

The success of the lesson will depend on you and your class. You know your students' strengths and weaknesses better than anyone else. Each lesson plan is there to help you to think about how to conduct a lesson in its chosen topic area. It is not a prescription of what to do.

11.1 Foundation

The circular functions

Objectives

- Solve right-angled triangles using the standard trigonometric ratios.
- Use Pythagoras' theorem to find the distance between two points in the Cartesian plane.
- Set up the Cartesian equation of a circle.
- Use the unit circle $x^2 + y^2 = 1$ to plot graphs of the circular functions sin $\theta°$ and cos $\theta°$ for any angle $\theta°$, where $0° \leq \theta° \leq 360°$.
- Know that any point on the unit circle has coordinates (cos $\theta°$, sin $\theta°$), where $\theta°$ is the angle the radius to the point makes with the positive x-axis
- Use Pythagoras' theorem to show that $\sin^2 \theta° + \cos^2 \theta° = 1$ for any angle $\theta°$.

Starter

Vocabulary

acute angle
obtuse angle
reflex angle
sine
cosine
hypotenuse
adjacent side
opposite side
Pythagoras' theorem
radius of a circle
centre of a circle
equation of a circle

Resources

Mini-whiteboards

Begin with a few review questions. Emphasise the use of appropriate vocabulary.

Q In a right-angled triangle, what is the name given to the longest side?

Q What is an acute angle?

Q Given an acute angle in a right-angled triangle, how is the sine of that angle calculated? What is the cosine of that angle?

Q What is an obtuse angle? What is a reflex angle?

Q If one of the acute angles in a right-angled triangle is θ, what is the other acute angle?

Q What is Pythagoras' theorem?

Q How can this theorem be used?

Q What is the length of the third side of a right-angled triangle in which the hypotenuse has length 5 cm and one of the other sides has length 4 cm?

Q What important property characterises a circle?

Q How can Pythagoras' theorem be used to set up the equation of a circle of radius r with centre at the origin? Write down the equation of this circle.

Main activity

Vocabulary

unit circle
first, second, third and fourth quadrants
$\cos^2 \theta$
$\sin^2 \theta$
circular function
domain
range
sinusoidal

Explain to the class that the theme of the lesson is to investigate the description of movement around a circle in order to introduce a set of functions known as circular functions and to establish some properties of these functions.

Show **OHT 11.1a**. Ask the class to describe precisely what is shown on the slide. Establish that the slide shows a circle of radius 1 unit, centred at the origin of coordinates. Ask students:

Q How can the equation of this unit circle be established?

Steer the discussion if necessary towards suggesting labelling a general point P on the circle with coordinates (x, y) and then using Pythagoras' theorem to set up the equation defining the circle in terms of x and y.

Q Does it matter where on the circle the general point P is placed?

Q If the y-coordinate of P is $^1/_2$, what is the exact value of the x-coordinate of P?

Q If the x-coordinate of P is $^1/_2$, what can be said about the y-coordinate of P?

Resources

OHTs 11.1a–11.1c

Resource 11.1b, one per student

Graph paper

Graph plotting software or graphics calculator

Protractors

Explain that as P moves around the circle in an anticlockwise direction, starting from the point (1, 0) on the x-axis, it moves successively through the first, second, third and fourth quadrants.

Q If P is in the first quadrant, what can be said about the angle that the radius OP makes with the positive *x*-axis?

Q What happens to this angle as P moves into the second quadrant and then in to the third or fourth quadrants?

Q In which sense, clockwise or anticlockwise, is this angle increasing as P moves into each quadrant?

Ask students to tabulate whether the x- and y-coordinate values are positive, negative or zero as P moves through the four quadrants in order.

Now show **OHT 11.1b** and point out the right-angled triangle. Ask students to sketch a copy of the figure and write next to each side of the triangle the length of that side. Then ask:

Q Is the labelling in this slide consistent with what you have written for the length of each side and with what you know about the trigonometry of right-angled triangles? Explain your reasoning carefully.

Explain that Pythagoras' theorem can be applied to the triangle OPN to give the result:

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \qquad (1)$$

Observe that this is conventionally written as:

$$\sin^2\theta + \cos^2\theta = 1 \qquad (2)$$

and that $\sin^2\theta$ is read as *sine-squared theta* and that $\cos^2\theta$ is read as *cos-squared theta*.

Now ask students to suggest how to generalise the picture shown in OHT 11.1b, using questions like:

Q What changes, and what stays the same, when triangle OPN is reflected in the *y*-axis? What happens when it is then reflected in the *x*-axis? What happens when it is then reflected in the *y*-axis?

Q Does it make sense to always call the *x*-coordinate of P *cosine θ* and its *y*-coordinate *sine θ*? Explain your answer carefully.

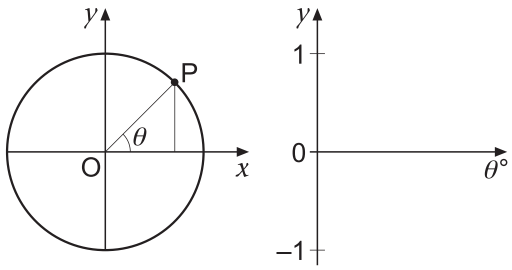
Bring out that the concepts of $\cos\theta$ and $\sin\theta$ are not confined to a right-angled triangle (although that is context in which they were first introduced) and that these functions can be used respectively as the x- and y-coordinates of *any* point on the unit circle. Also point out that result (2) will always hold, no matter in which quadrant P is positioned, and that this result is simply a special statement of Pythagoras' theorem.

Tell students that they will now investigate how the coordinates of P change as P moves round the circle in an anticlockwise direction, starting at the point (1, 0).

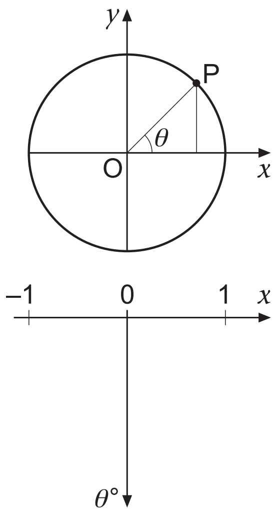
Give each student **Resource 11.1b** and two sheets of blank graph paper. Ask students to:

- Mark on Resource 11.1b the positions of P every 15° as P moves around the circle.

- Position Resource 11.1b with the picture to the left of one piece of graph paper. Draw on the graph paper a y-axis parallel to the y-axis in the picture and calibrated in the same scale, and a θ-axis, parallel to the x-axis in the picture, calibrated from 0 to 360 degrees (see below).



- Next, position the picture on Resource 11.1b so that the x-axis and the θ-axis lie along the same line. Project across the y-value of each position of P to the relevant angle on the θ-axis to give a series of points that can be joined by a smooth curve.



Q What is the equation of this curve for y in terms of θ?

Q How should the exercise be adapted to show the variation of the x-coordinate of P as θ varies? (place the second piece of graph paper below the circle diagram with its θ-axis on the same line as the y-axis and with its x-axis parallel to, and calibrated in the same scale as, the x-axis in the circle figure)

Q What would be the equation of this variation?

Explain that the two functions generated in this way are called *circular functions* and show **OHT F11.1c** to illustrate the correct drawings of the graphs of these two circular functions. Ask students simple questions about the two graphs:

Q Can you suggest a way of moving one graph into the other?

Q Does the range of these functions have to be restricted to vary between 0° and 360°?

Q What do you think the negative angles on the scale mean?

Q What is the range of each of the graphs?

If possible, show the class a dynamic illustration of how these graphs are generated using graph plotting software or the trace facility on a graphics calculator.

Demonstrate how one of the functions may become the other by a constant shift of the angle by 90° one way or the other.

Say that this wavy motion is often described as *sinusoidal*.

Further work

Interesting extension work can be vividly brought to life using an interactive graphing software package, but the work can also be carried out with a graphics calculator.

Students could be asked to:

- investigate the graph of the function $y = \tan x$, where $\tan x = \sin x/\cos x$;
- explore the effect of working on a circle of radius A and the effect of setting the zero of the angle θ at a different position;
- explore the effect of doubling or halving the argument of the function, i.e. the definition of the angle being measured.

Consolidation

Ask students a few questions to check on what they have learned.

Q What is the interpretation of the circular function $y = \sin \theta$?

Q What is the interpretation of the circular function $x = \cos \theta$?

Q What is the relation between sine and cosine which is a special form of Pythagoras' theorem?

Q What is the equation of the unit circle centred at the origin of coordinates?

Q What are two characteristic features of the graphs of the sine or cosine functions?

Q In which quadrants are the sine values positive? In which quadrants are they negative?

Q In which quadrants are the cosine values negative? In which quadrants are they positive?

Q What angles give zero for the values of the sine function? What angles give zero for the values of the cosine function?

Summary for students

- The equation of the unit circle centred at the origin is $x^2 + y^2 = 1$.
- The coordinates of any point P on the unit circle are ($\cos \theta$, $\sin \theta$), where θ is the angle the radius to P makes with the positive x-axis.
- The anticlockwise direction of rotation is used to measure positive values of θ (and a clockwise rotation is used for negative values of θ).
- The identity $\sin^2 \theta + \cos^2 \theta = 1$ is true for all values of θ, and is a special case of Pythagoras' theorem.
- The graph of $y = \sin x$ varies so that $-1 \leq y \leq 1$; $y = 0$ when $x = 0°$, 180° or 360°.
- The graph of $y = \cos x$ varies so that $-1 \leq y \leq 1$; $y = 0$ when $x = 90°$ or 270°.
- Each of the above function can be made into the other by a shift in the angle.
- Each of these functions represent a sort of 'wave' and the manner in which they vary is called *sinusoidal*.

11.2 Foundation

Working with quadratic functions

Objectives

- Recognise a second order polynomial in one variable as a quadratic function; plot graphs of such functions and pick out the intercepts with the coordinate axes, the axis of symmetry and the coordinates of the maximum or minimum point.
- Understand when such functions are increasing, when they are decreasing and when they are stationary.
- Solve quadratic equations exactly, by factorisation, by completing the square and by using the quadratic formula.
- Understand simple quadratic inequalities.
- Model a range of situations with quadratic functions.
- Translate the statement *y is proportional to* x^2 into the symbolism $y \propto x^2$ and into the equation $y = kx^2$; know that the graph of this equation is a parabola through the origin.

Starter

Vocabulary
quadratic function
parabola
axis of symmetry
maximum
minimum
stationary point
y-intercept
quadratic equation
solution set

Resources
OHT 11.2a

Inform students that this lesson focuses on quadratic functions, their graphs and associated quadratic equations. Start with some quick recall questions, showing **OHT 11.2a** when needed.

Q What is a quadratic function? Write down three different quadratic functions.

Q What is the general form of a quadratic function in one variable? ($y = ax^2 + bx + c$)

Q What is the name of the curve that represents a quadratic function of one variable? (a parabola)

Q The top part of OHT 11.2a shows two quadratic graphs. What is the equation of each graph?

Q The bottom part of OHT 11.2a shows another quadratic graph. What is the equation of its axis of symmetry? What are the coordinates of its maximum point? What is its *y*-intercept?

Q In what subdomain of *x* does the function in this picture increase? At $x = 4$ is the function increasing or decreasing?

Q What is the solution set of the quadratic equation $(x - 3)(x + 2) = 0$?

Q Consider the function $f(x) = 4 + (x - 2)^2$? What is the equation of the axis of symmetry of the function? Does this function have a maximum or a minimum point? What are the coordinates of its stationary point?

Main activity

Vocabulary
trajectory

Resources
OHT 11.2b

Continue the lesson with a series of illustrative examples showing how quadratic functions can be used in real-world applications. Start the discussion by asking students to imagine that they are throwing a baseball as far as they can.

Q What does the trajectory of the ball look like?

Make sure that all eventually agree that the path is a parabola.

Q Can you think of other situations that might be represented by parabolic motion? (jets of water from a fountain; a bullet shot from a gun)

Q What mathematics would be appropriate to model these situations?

Now work through an example with the class. Describe the situation:

> *A girl throws a baseball. At the moment of the throw the ball is 2 m above the ground, which is horizontal. The baseball just goes over a wall of height 4 m that is 20 m away from the girl. When the ball hits the ground it is 30 m from the girl.*

Explain that if the equation of the trajectory can be worked out then it can be used to work out how far from the girl the ball is at any height, or how high the ball is when it is a given horizontal distance away from the thrower.

Get the class to generate ideas as to how the physical problem might be turned into a mathematical model that can be analysed and used to make predictions.

Q Since we have agreed that the path of the ball is a parabola, what generalised function can be used to start to model the mathematics of the situation? ($y = ax^2 + bx + c$)

Q In this model, what do the variable quantities represent? (horizontal and vertical distance from the thrower, or from some other point of reference)

Q The model uses Cartesian coordinates. From where would it be sensible to set up coordinate axes? (there are two sensible choices, each with horizontal and vertical axes: where the girl is positioned on the ground or the point at which the ball goes into flight)

Q What pieces of information can be used to find out more about the mathematical model and to make it more particular to this situation?

Q What do you notice about the way in which the numerical information has been given? (the information, either directly or implicitly, links a vertical height to a horizontal distance)

Tell students to cast their minds back to when they solved two simultaneous linear equations.

Q How can the crucial information be used to set up and solve a pair of simultaneous linear equations with two unknowns?

Work with the origin at the girl's feet on the ground. (Imagine that her feet are at a single point on the ground. Explain that such simplifying assumptions are often necessary to get started with a mathematical model, and that complications can be introduced at a later stage, if required.) Try to lead the discussion so that students generate the relevant equations:

$$4 = 400a + 20b + c, \text{ since } y = 4 \text{ when } x = 20 \qquad (1)$$
$$0 = 900a + 30b + c, \text{ since } y = 0 \text{ when } x = 30 \qquad (2)$$
$$2 = c, \text{ since } y = 2 \text{ when } x = 0 \qquad (3)$$

Allow students time to solve these equations and ask them to volunteer their solutions.

Confirm that the solution of the three equations is $a = -{}^{1}/_{60}$, $b = {}^{13}/_{30}$ and $c = 2$.

Q What is the mathematical equation that describes the flight of the baseball? ($y = -\frac{1}{60}x^2 + \frac{13}{30}x + 2$)

Q How can this equation be simplified so that it does not contain any fractions? (multiply throughout by 60)

Q What is the simplified equation? ($-x^2 + 26x + 120 = 0$)

Now ask the class to imagine that the ball could go backwards in time from the moment that it was thrown. There would then have been some earlier time at which the ball was level with the ground.

Q Can we work out how far away from the girl the ball would have been in this imaginary situation? (yes, by solving the quadratic equation $-x^2 + 26x + 120 = 0$)

Ask the class to verify that the two solutions of this equation are $x = 30$ and $x = -4$. Point out that the solution to the present problem is the negative solution (backwards in time and in distance) and that $x = 30$ is the solution that they already knew about.

Q Now that we know the two positions of the ball at ground level, can we work out the axis of symmetry of the parabolic path? (yes, by finding the x-value halfway between the two extreme values at ground level)

Q What is the equation of the axis of symmetry? ($x = 13$)

Q How can the maximum height of the ball be calculated? Calculate the maximum height exactly and then round it to two decimal places. (substitute $x = 13$ and calculate y, which gives the maximum height as $4^{49}/_{60}$ m, or 4.82 m to 2 d.p.)

Now ask students if they would expect any different answers if the origin of coordinates had been taken at the point at which the ball went into flight. Get them to suggest what might be the same, and what might be different, when this coordinate system is used.

Tell the class that they will now look at another problem that involves modelling with quadratic functions. Show the illustration on **OHT F11.2b**. Explain that it depicts the central section of a suspension bridge across a wide expanse of water. Ask students if they have ever seen a suspension bridge or if they can name some famous ones around the world.

Inform students that AB represents the road, QP and RS are frameworks that hold the bridge up together with a very thick and strong steel cable that is attached to the frameworks at P and R and supports the structure on which the road is built. The actual form of the cable (when it is suspended from two points) is called a *catenary* (meaning ‘chain’), but a very good approximation to a catenary is a parabola.

Q If the shape of the cable is a parabola, what mathematical model can be used to describe the curve taken up by the cable?

Q Using this model, where would be a good place to set up the origin of coordinates? (at the lowest point of the cable, taking the road as the x-axis and the vertical through the lowest point as the other axis in order to make the parabola symmetric with respect to the x-coordinates)

Now show the lower part of OHT F11.2b, which contains some information about the bridge and some problems to solve. Engage students with questions about how they might attempt to answer the various questions about the bridge. Then allow time for them to do the questions and to check the solutions.

Further work

Resources

Graph plotting software

Further work on quadratic functions could explore the behaviour of these functions and their equations as their curves are moved parallel to either axis, and as the axis units are rescaled.

This is best done using dynamic graphing software so that students see for themselves how the curves change with translations in the direction of either axis and with rescaling along either axis.

Other work could be to investigate more applications that can be modelled by quadratic functions, for example, rolling a ball diagonally across an inclined plane, or looking at a body moving with constant acceleration to see how distance moved depends on the time taken.

Consolidation

Bring the class together to ask some snappy questions about what they have learned.

Q What class of functions have we been studying?

Q What types of curves are exhibited by these functions?

Q What are some of the key features of these curves?

Q What is the general Cartesian form of these functions?

Q Give some examples of situations that can be modelled using quadratic functions.

Summary for students

- Quadratic functions have the general form $y = ax^2 + bx + c$. The graphs of these functions are all parabolas.
- Every parabola has an axis of symmetry and either a maximum point or a minimum point.
- If one quantity y is proportional to x^2 its Cartesian equation has the form $y = kx^2$, where k is a constant.
- The graph of $y = kx^2$ is a parabola through the origin, with its maximum or minimum point at the origin.
- Examples of phenomena can that can be modelled by quadratic functions include the trajectories of balls or jets of water from a fountain, and the cables of suspension bridges.

11.3

Advanced

Probability of single and combined events

Objectives

- Understand when two events are mutually exclusive, and when a set of events is exhaustive.
- Know that the sum of probabilities for all outcomes of a set of mutually exhaustive and exclusive events is 1.
- Know that when two events A and B are mutually exclusive the probability of A or B is $P(A \cup B) = P(A) + P(B)$.
- Know that two events A and B are independent if the probability of A and B occurring together is $P(A \cap B) = P(A) \times P(B)$.
- Use tree diagrams to represent and calculate the probabilities of compound events when the events are independent and when one event is conditional on another.
- Know that in general if event B is dependent on event A, then the probability of A and B both occurring is $P(A \cap B) = P(A) \times P(B|A)$.

Starter

Explain that the lesson is mainly about the probabilities of combined events, but that it is necessary to first review the probabilities associated with a single event.

Q Between what two values does the probability of a single event lie? What do the extreme values mean?

Q If P(A) is the probability of the occurrence of event A, what is the probability of event A not occurring?

Tell students that this is the *complementary probability*, and that if P(A) is the probability of the event A occurring then the probability of A not occurring is denoted by $P(A')$, with $P(A') = 1 - P(A)$.

Q A many-sided die is thrown. Each face has a different number on it. The probability of throwing a 3 is $^1/_{12}$. What is the probability that the upper face is not 3?

Q What is a random variable?

Q What is the sample space of a random variable?

Q A coin is tossed 7 times and the number of tails is counted. What is the sample space? (there are eight possible outcomes, {0, 1, 2, 3, 4, 5, 6, 7})

Q What is the probability of all possible outcomes for a given sample space?

Main activity

Vocabulary

event
exhaustive
mutually exclusive
independent
complementary probability
conditional probability
tree diagram
union
intersection
sample space
Venn diagram

Resources

OHTs 11.3a and 11.3b
Resource 11.3b, one per student

Describe the rest of the lesson as an exploration of the mathematics of the probability of two or more events.

Start with an example to illustrate the concept of two *mutually exclusive* events, also sometimes known as *disjoint* events. Count the number of students in the class, and the number wearing spectacles. Record these numbers on the board.

Q What is the probability that a student in class chosen at random is wearing spectacles? What is the probability that a student chosen at random is not wearing spectacles? Are there students who are not either wearing spectacles or not wearing spectacles? What is the sum of the probabilities of choosing a student that wears spectacles or does not wear spectacles?

When all agree that the answer is 1, emphasise that a probability of 1 means certainty: it is certain that any student picked at random will either be wearing spectacles or will not be wearing spectacles. The event 'wearing spectacles' cannot happen simultaneously with 'not wearing spectacles', as at the moment of selection any student is or is not wearing spectacles. This is what is meant by mutually exclusive.

Now ask the class to consider the situation in which there are 25 balls in a bag; 12 of these are red, 5 are green and the rest blue. A ball is drawn at random from the bag.

Q How many mutually exclusive events are possible when drawing one ball from the bag? What is the probability of each event?
What is the probability of drawing either a red or a green ball?
What is the probability of drawing either a blue or a green ball?
What is the probability of drawing a ball of any of the three colours?

Q What is the probability of not drawing a red ball? Describe two ways in which this can be calculated.

Q What do you think is the rule for the probability of one mutually exclusive event or another?

This will lead into the probability rule for two mutually exclusive events A and B. First establish the notation P(A) for the probability of event A, P(B) for the probability of event B and $P(A \cup B)$ for the probability of event A or B, justifying the latter notation (union of two sets) with reference to a Venn diagram. Then give the 'addition rule' rule for P(A or B):

$$P(A \cup B) = P(A) + P(B) \qquad (1)$$

This rule can be extended to three mutually exclusive events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \qquad (2)$$

and so on.

Explain that if the mutually exclusive events cover the entire sample space then they are said to be *exhaustive* and that the sum of the probabilities of all the mutually exclusive and exhaustive events is 1, since one of these events is certain to happen.

Now introduce the concept of *independent events*, explaining that, with two independent events, knowing that one event occurs does not change the probability that the other event occurs. Each event has no 'memory' of the other. Ask students to suggest situations with two independent events. Then lead a discussion to try to get students to develop the mathematics for the probability of two independent events occurring together, or simultaneously. The notation for the probability of two events A and B occurring together again draws on set theory notation (intersection of two sets), and the rule here for P(A and B), when A and B are independent, is the 'multiplication rule':

$$P(A \cap B) = P(A) \times P(B) \qquad (3)$$

Explain to students that often the probability assigned to an event can change if another event has occurred. For example, there is a probability in each country that a woman will live to at least 80 years of age, but if we know that that woman comes from a family that has suffered from genetic heart disease, then her probability of reaching 80 is considerably diminished. When one event depends on another event in this way, the probability of its occurrence depends on the probability of the other event having occurred. The notation P(B|A) is used to denote the probability of B given that A has occurred, and is called a *conditional probability*. When B is dependent on A in this way, the multiplication rule gets generalised to

$$P(A \cap B) = P(A) \times P(B|A) \qquad (4)$$

This leaves (3) as a special case of (4) when P(B|A) = P(B), which is the condition that B is independent of A.

Illustrate rule (4) by showing **OHT 11.3a**. Allow the class a few minutes to work out some other conditional probabilities and compound probabilities associated with the population table in the OHT.

Point out that (4) can be rearranged so that the conditional probability can be calculated as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad (5)$$

Q What is the probability that a woman chosen at random from the population described in OHT 11.3a is over 61?
What is the probability that the woman is over 61 and divorced?
Using result (5), what is the probability that the woman has been divorced given that she is aged 61 or more?
Check your answer by an alternative method.

Continue the lesson by introducing the notion of a *tree diagram*. Use an OHT of **Resource 11.3b** to illustrate the situation where coloured balls are drawn at random from a bag and not replaced. The bag originally contains 12 red balls, 4 yellow balls, 3 blue balls and 1 white ball. Give each student a copy of Resource 11.3b. Discuss with students how to calculate the probability that at least three yellow balls have been drawn out of the bag after four balls have been removed. Now get students to write down, on their own copies, the probabilities on each branch of the tree and then to calculate the required probability.

When they have done this, explain that the tree diagram is a very useful form of representation that combines both the addition law for probabilities (rule 1 and its generalisation) and the multiplication law (rule 4 and its generalisation). Each point at which branching occurs is one stage of the problem, and the probabilities at each of these points sum to 1.

Q Why?

Probabilities (some of which, as in this case, are conditional probabilities) are multiplied along any one complete branch of the tree and the desired outcomes are summed for all the complete branches that give these outcomes.

For the problem with the balls removed from the bag there are five branches that lead to the final outcome of at least three yellow counters after four removals without replacement:

$$\text{P(Y and Y and Y and Y)} = \tfrac{4}{20} \times \tfrac{3}{19} \times \tfrac{2}{18} \times \tfrac{1}{17} \approx 0.0002 \text{ and}$$
$$\text{P(Y and Y and Y and } \tilde{\text{Y}}) = 4 \times \tfrac{16 \times 4 \times 3 \times 2}{20 \times 19 \times 18 \times 17} \approx 0.0132$$

so that

$$\text{P(at least three Ys) = P(four Ys) + P(Y and Y and Y and } \tilde{\text{Y}}) \approx 0.0134$$

Q What is the probability that no more than 2 yellow balls are removed when four balls are taken from the bag? What is the quickest way to calculate this probability?

End the main part of the lesson by codifying the rule implicit in the probability tree. The generalisation of the multiplication law is to condition each event on the occurrence of all previous events. Thus

$$\text{P(A and B and C)} = \text{P(A)} \times \text{P(B | A)} \times \text{P(C | A and B)} \qquad (6)$$

and so on.

Further work

Vocabulary
Bayes' theorem

Further work could be set on generalising the addition law to include cases where two events can occur simultaneously.

Harder extension work could explore the idea of running backwards through a probability tree. For example, using Resource 11.3b one might ask what is the probability that in a YYY$\tilde{\text{Y}}$ outcome the $\tilde{\text{Y}}$ was picked at the third draw. This will lead to the important result known as Bayes' theorem. Named after an English clergyman who discovered this result in 1763, it has profound consequences in medical statistics and in the study of risk in insurance and in finance.

Consolidation

Ask a few quick questions to check what students have learned:

Q If P(A) is the probability of event A occurring, what is the probability of event A not occurring? What name is given to this probability?

Q What is meant by two events being: mutually exclusive? independent?

Q What is a conditional probability?

Q How is P(A and B) calculated? How is P(A or B) calculated?

Summary for students

- The complementary probability of P(A) is P(A′) = 1 – P(A).
- The probability of either of two mutually exclusive events is known as the addition rule: $P(A \cup B) = P(A) + P(B)$.
- The probability of two events A and B is $P(A \cap B) = P(A) + P(B|A)$.

  The probability of B given that A has occurred is $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.
- A tree diagram is a useful representation of both the multiplication rule and the addition rule, especially where information is difficult to see at a glance.

11.4 Advanced

Proof

Objectives

- Develop chains of logical reasoning, using correct mathematical notation and terms.
- Explain their reasoning, both orally and in writing.
- Generalise wherever possible.
- Generate mathematical proofs, and identify exceptional cases.

Starter

Announce that the lesson is about mathematical proof in its various manifestations, and that it will bring together many ideas and topics that have been studied throughout the course so far.

You may wish to read the following quote to the class.

> *The man who discredits the supreme certainty of mathematics is feeding on confusion, and can never silence the contradiction of sophistical sciences, which lead to eternal quackery. (Leonardo da Vinci, 1452–1519)*

Start with a few quick-fire questions.

Q What is special about proof in mathematics?

Q Why is proof considered so important in mathematics?

Q What different methods of proof in mathematics can you think of?

Q What is wrong with the following reasoning?

If $2 = 1$
then $1 = 2$
and adding each equation gives
$3 = 3$
This is true, so $2 = 1$.

Q Is $n^2 + n + 17$ always prime when *n* is a positive integer? (no; choose *n* =17 for example)

Q Is $\sqrt{2}$ rational? How do you know? Is $1 + \sqrt{2}$ rational?

Q What is the largest prime number? How do you know?

Q What is the smallest prime number? Why?

Q List some proofs already encountered in the course.

Q If A implies B, does B imply A? Explain your reasoning.

Q Can you prove that all numbers are interesting?

Main activity

Vocabulary

true, false
undecidable
proof
hypothesis
conjecture
premise
proposition
axiom
postulate
definition
deductive logic
conclusion

Resources

OHTs 11.4a, 11.4b

Now prove to students that all numbers are interesting by showing **OHT 11.4a**.

Begin by emphasising that there is no other subject that can make statements with the certainty of mathematics, and that this is the source of its power and fascination. Talk around the following.

Statements in mathematics are *true* or *false* or *undecidable*. No theory in physics, or in any other science, can be proved to be true in the sense that Pythagoras' theorem, for example, can be proved in Euclidean geometry.

In the sciences, a good theory is likely to be true but can never be verified absolutely because it is not possible to carry out every possible measurement for all time to prove the theory. The best one can hope for is a series of experimental results that do not cast doubt on the theory. So in the sciences a theory can be proved false but can never be verified absolutely.

In the humanities, most statements are a matter of opinion, and chains of reasoning can be disputed by arguing from a different set of premises.

There is much more agreement in mathematics about what constitutes the basic set of premises in different areas of the subject, and the subject itself is organised in such a way as to lead to maximum internal self-consistency. Proof is the essence of mathematics, because it is through proof that some mathematical statements are true and others are false. Mathematics builds on statements that are true in order to establish yet more statements that are true. It is forever doing this. It is this too that makes mathematical modelling such a powerful analytical tool for exploring other disciplines such as physics, electronics, engineering or medicine.

Build a discussion around the question:

Q What is a proof?

Make sure that the discussion includes the question of rigour and the idea of convincing an audience, and that differing degrees of rigour might be acceptable to different types of audience.

Suggest to the class some words that might be relevant when trying to fix on a result to prove. The list could include words like:

hypothesis *conjecture* *proposition* *premise*
axiom *postulate* *definition*

Ask students to volunteer meanings, or examples, of these words.

Stress that the process of proving statements in mathematics relies on the use of deductive logic in which a conclusion follows necessarily from a premise; if the premise is true then the conclusion is also true. Words in logic that are useful in attempting a proof are:

implication *equivalent* *a necessary condition* *a sufficient condition*
a necessary and sufficient condition *if and only if*

Get students to suggest examples here, and use the appropriate notation to illustrate them. Use **OHT 11.4b** as a back-up for examples.

Discuss with students different methods of proof in mathematics. Work through some proofs with students, getting them to suggest the successive steps in the proof. This will probably take *two* sessions.

Common methods of proof, together with examples, include the following.

Proof by contradiction To prove result P, the idea is to show that its negation P′ (that is not-P) is false.

1 Prove that $\sqrt{2}$ is irrational.

2 Prove that every prime number greater than 3 can be written as $6n \pm 1$, where n is a positive integer.

Proof by exhaustion In this method of proving all possibilities are considered.

3 Prove that the only regular polygons that tessellate the plane have 3, 4 or 6 sides.

Proof by deduction Mathematics is a deductive science. Deduction is made explicit in *geometry* where proofs are deduced from a basic set of axioms, and also from other results already established.

4 Given that the angles on a straight line sum to 180°, prove that the angles in a triangle sum to 180°.

5 Prove that if two chords AXB and CXD of circle ACBD intersect at X, then AX × XB = CX × XD.

6 Prove Pythagoras' theorem.

Deduction is also used in *algebraic proofs* where the method is to go from step to step using the rules of algebra as the basic axioms, and with each step implied by the previous step.

7 Prove that the product of two numbers is even if and only if at least one of the numbers is even.

8 Prove that if five consecutive integers are squared then the mean of their squares is 2 more than their median value.

9 Prove that $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{a+c}{b+d}$, stating carefully the conditions under which the result holds.

10 Prove that a triangle with sides $2n$, $n^2 + 1$ and $n^2 - 1$ is a right-angled triangle. Is the converse true?

Another use of deduction is to prove *trigonometric identities*.

11 Prove that $\cos^4\phi - \sin^4\phi \equiv \cos^2\phi - \sin^2\phi$. [Note that the logic here must move from $\cos^4\phi - \sin^4\phi$ to the equivalent form $\cos^2\phi - \sin^2\phi$; it is incorrect to work with both forms simultaneously, or to assume the answer in what is to be proved.]

Disproving a general statement is often done by **disproof by counter-example**. A counter-example is a particular case that disproves a general proposition.

12 Prove that the statement *every number of the form* $6n \pm 1$ *is prime* is false.

Further work

Vocabulary

mathematical induction
Goldbach's conjecture

Resources

Resource 11.4c

Give students much practice in using logical deduction and at proving a range of results. The work should include filling in missing logical connectives between statements, correcting faulty logic, and providing their own proofs. A few suggestions are on **Resource 11.4c**. Questions 5 and 6 are more difficult than the others.

Interesting further work could be set by asking students to conjecture and prove certain results using **mathematical induction**, a recursive way of moving from particular statements to general mathematical conclusions. The essence of an inductive proof is to show that a statement P involving a variable positive integer value n is true for some positive integer n_0 and that if it is true for positive integer k it is then true also for the next positive integer $k + 1$. Logic then dictates that it is true for all positive integers $n \geq n_0$.

13 The sum of n terms of a series is defined by $s_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)}$. Substitute values of n to help guess a formula for s_n. Prove this formula by induction.

14 Prove that $11^n + 3\times 4^{n-1}$ is divisible by 7.

15 Use induction to prove that $(x+y)^n = \sum_{0}^{n} {}^nC_r x^{n-r} y^r$ for any positive integer n, where the binomial coefficient ${}^nC_r = \frac{n!}{r!(n-r)!}$ and $0! = 1$.

Further examples of induction are included as questions 5 and 6 on Resource 11.4c.

Students could be set a challenge to research and attempt to solve a famous unsolved problem known as Goldbach's conjecture: *Every even number greater than 2 can be written as the sum of two primes*. Any student that solves this problem will become famous overnight!

Consolidation

Go round the class asking students in turn to say one thing that they have learned in the lesson. No student should repeat what has already been said. If there are too many students to do this, then ask some simple questions like:

Q True or false? (a) $x^2 > 9 \Rightarrow x < -3$

(b) $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$ **(c)** $\theta = \frac{3\pi}{4} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

Q What is the negation of the statement *7 is a prime number greater than 5*?

Q What is disproof by counter-example? Give a simple instance of this.

Summary for students

- Proof is fundamental to all of mathematics.
- Common methods of proof in mathematics are proof by contradiction, proof by exhaustion, proof by induction and proof by deductive reasoning.
- Disproving a general statement is often done by disproof by counter-example.