Mathematics lessons for Grade 12

Lessons in this section

Using these lesson plans

Each of these sample lessons for Grade 12 is suitable for use with a whole class. The lessons are standalone examples to illustrate different teaching and learning activities. They are not intended to be taught as a sequence.

The first two lessons are based on the Grade 12 standards for foundation mathematics, the third on Grade 12 standards for advanced quantitative methods and the fourth on Grade 12 standards for advanced mathematics for science. The relevant standards are shown in the lesson plans.

The lessons are drawn from different topics to show spread rather than sequence. They illustrate different pedagogical styles for the various topics and the different sets of standards.

Each lesson has been organised in three parts: a starter to introduce the lesson, a main activity, and a final phase to help students to reflect on the lesson and consolidate their learning. Early in the lesson, you should outline the purpose of the lesson, drawing out for students what they will learn and how this builds on previous work. In the final part of the lesson, you will need to establish the key learning points, what students should remember and what they will go on to learn next about the topic. You should also help students to appreciate the links that can be made to other topics in mathematics, in other subjects or in the real world.

The lesson plan suggests how you as the teacher might teach the class and interact with students but you will need to add your own questions to present and develop the mathematical threads that run through the lesson. The plan makes no attempt to tell students how or when to take notes of the lesson.

The lesson plans do not include homework tasks because the lessons are single examples taken out of sequence. You will need to provide homework since it is an important part of a lesson.

There may be too much material in the lesson plan for one lesson (e.g. of 60 minutes), since the rate of teaching and learning will depend on the class. In this case, you might designate one of the activities in the lesson as homework, or carry it forward to the next lesson. Be selective about which activity to cut – it does not have to be the last one merely because it comes at the end.

You may need to supplement the activities in the lesson with simpler or more challenging tasks if the students in your class have a range of attainment. You could choose from activities in textbooks or from your own resources. If you wish, different tasks can be given to different groups of students, according to their needs. Each lesson plan includes some suggestions for further tasks.

Answers to some questions are provided to help you to correct students' responses and give feedback. Many of the questions are open-ended and have more than one possible answer. They are there to facilitate critical discussion, to help students to generalise and to help them gain insight. They illustrate style and a pedagogical approach rather than the words you should use yourself.

The success of the lesson will depend on you and your class. You know your students' strengths and weaknesses better than anyone else. Each lesson plan is there to help you to think about how to conduct a lesson in its chosen topic area. It is not a prescription of what to do.

12.1 Foundation

Modelling growth

Objectives

- Generate recursive sequences from term-to-term and position-to-term definitions to model the behaviour of real-world situations.
- Understand exponential growth and decay and the forms of the associated graphs.
- Use a graphics calculator to plot the exponential function e^x and the natural logarithm function $\ln x$; know that one is the inverse function of the other.
- Solve for x the equation $y = a^x$ and use this in problems.

Starter

Vocabulary

the exponential function
natural logarithm function
inverse function
recurrence relation

Resources

OHT 12.1a
Graphics or scientific calculator

Ask a series of questions to remind the class about the special number 'e', about the exponential function $y = e^x$, and about recurrence relations.

Q What are the characteristics of an exponential curve?

Q What is the inverse function of the exponential function?

Q How are the graphs of a function and its inverse function related to each other?

Show the students unlabelled graphs of $y = e^x$, $y = e^{-x}$ and $y = \ln x$ (**OHT 12.1a**). Ask them to identify each curve.

Q What is the natural logarithm of $e^{0.12}$?

Q How can the equation $8 = 5e^{4x}$ be solved to find x?

Ask students how to use their calculators to calculate values of e^x and $\ln x$. Get them to calculate specific values of these functions when x is given.

Q What is a recurrence relation? Invent an example of a recurrence relation.

Q What is the value of u_4 given the recurrence relation $u_n = 3u_{n-1}$ with $u_0 = 0.5$?

Main activity

Vocabulary

mathematical model
population
capital
exponential growth
decade
recursive
sequence
discrepancy

Resources

Graphic calculator or computer with spreadsheet and graph plotting software
OHT 12.1b

Tell students that in this lesson they will study growth of population or of capital and related phenomena that are best described by similar mathematical models.

Begin with a problem about the growth of the population of the United States of America. The problem is analysed from different points of view to give different insights into the modelling process and to help make connections between different aspects of mathematics.

In 1790, the population of the United States of America was 3 929 214. After analysing the population over the next few decades a recursive model was proposed in which $P_n = 1.24 P_{n-1}$, where n is the number of decades from 1790.

Q What does P_0 mean? What is its value? (P_0 is the population in 1790, namely 3 929 214)

Q What population does the model give for the year 1800? (4 872 225) **And for 1810?** (6 041 559)

Q Will these populations be the same as the actual populations in these years? Give reasons for your answer.

Q What does P_n mean? (P_n is the population n decades after 1790, where n is a positive integer)

Q What is the percentage increase in population each decade predicted by this model? (24%)

Q What type of sequence is the recurrence relation $P_n = 1.24\, P_{n-1}$? (a geometric sequence)

Demonstrate the growth of population using a spreadsheet or graphics calculator to generate the successive values of P_n. (If the class is familiar with using spreadsheets this could be set as a class activity.)

Encourage students to comment on the growth of the population in the period 1800–1900 and in the period 1900–2000.

Q In roughly how many decades does the population double its value? (3, or more accurately 3.2)

Use the model to predict the population in 2050.

Q How many times bigger is the predicted population for 2050 than the actual population in 1790? (268.5 times bigger)

Q What factors could be responsible for such a huge increase in population in these 260 years?

Now ask students to suggest a mathematical formula that predicts the population n decades after 1790.

Next, ask them to plot the graph of population growth against time, using a graphics calculator or graph plotting software.

Get the students to use the graph to find the times when the population first doubles in size, then doubles again, then doubles again and so on.

Explain that this population model demonstrates exponential growth.

Show **OHT 12.1b**. Inform the class that this shows the actual population of the United States for each decade from 1790 to 2000.

Now ask students to use a graphics calculator or graph plotting software to plot the actual population against time on the same axes as for the predicted population growth from the recursive model.

Q How accurate is the model compared to the real situation? Are there periods of time when the model is a good predictor of the actual growth pattern? (until 1970 the population in the USA grew faster than that predicted by the model; from 1980 the rate of population growth slows down and the model over-predicts the population)

Explain that the discrepancy between the model and the real situation at any time is the difference in values between the actual and the predicted population at that time.

Q What might be some factors that explain the discrepancies between the model of population growth and the actual population growth?

Now look at an analogous problem: the growth of capital in an investment savings account. Two banks pay interest on savings accounts, but each bank has a different approach to adding the interest. In bank A interest of 6% is added annually. In bank B interest is added n times per year at a rate each time of $6/n$ %. Assume that QR 10 000 is invested in each bank.

Q What mathematical expressions give the amount in each bank account at the end of one year? ($10\,000[1 + 0.06]$ and $10\,000[1 + 0.06/n]^n$)

Check these expressions and discuss equivalent solutions with the students.

Bank B now decides to add the interest more and more often during the course of the year and increases this frequency without limit. It can be shown that the mathematical model that can be used to calculate the total amount in the account at the end of one year is given by the formula:

$$P = 10\,000\,e^{0.06}$$

After n years the amount in this account is:

$$P_n = 10\,000\,e^{0.06n}$$

Explain that there are two ways to use this formula.

To calculate the amount after a certain number of years, simply replace n by that number and use the calculator to calculate the value of the expression.

Q How much is in the account at the end of 2 years? After 3 years?

The second way to use the formula is to ask the inverse problem: How long will it be until there is so much in the account?

Suppose we wish to find how long it will take to double the value of the initial investment.

Q What is the equation that has to be solved?

The equation to be solved is then $20\,000 = 10\,000\,e^{0.06n}$.

Q What is the value of $e^{0.06n}$? (2)

Q How can the equation $2 = e^{0.06n}$ be solved? Can it be solved graphically? If so, how? Can it be solved mathematically? If so, how?

Now introduce natural logarithms and show how to use a calculator to calculate them. Explain the mathematical solution: $\ln 2 = 0.06n$.

Q How can this equation be rearranged to find the value of n?

Q What is the value of ln 2 correct to four decimal places? (0.6931)

Q In how many years will the original investment double its value? Give the answer correct to three significant figures. (11.6 years)

Comment: If the investor had deposited the savings in bank A instead of in bank B it would have taken 11.9 years to double the value of the savings, so bank B is the better bet, but only marginally so.

Now get the class to generate and solve similar problems on their own. Discuss what features they can vary and what structures need to remain fixed to keep to the same type of problem.

Extension

Further work could generalise the concept of exponential models.

Many situations exhibit exponential behaviour. Here are a few examples that could be explored further. There are many more!

- In music, harmonic scales have an exponential behaviour governing the frequencies of the notes. The frequency of vibration of each string in a piano is proportional to the length of the string. A grand piano has an exponential shape as its profile!

- Sound is measured in decibels, which is a logarithmic scale.
- Earthquakes are measured on the Richter scale, which is another logarithmic scale.
- Radioactivity is measured using exponential decay functions of the form $y = Ae^{-kx}$. Exponential decay functions are mirror images in the y-axis of corresponding exponential growth functions. Radio-carbon dating is a further application of this exponential decay model. The analogous concept to doubling time in exponential growth is the concept of half-life for exponential decay (see lesson 12.4).
- Cooling of a heated object follows an exponential fall-off of temperature over the temperature of the surroundings.
- The amount of medication left in the body a certain time after taking the medication also follows an exponential fall-off.

Consolidation

Say that a feature of the lesson has been the study of exponential growth. The situations may differ, but the underlying mathematics is the same.

Ask students to form small groups to record two important features of what they understand by exponential growth and to suggest their own examples of exponential growth. Bring the groups together to compare ideas.

Now ask a few quick questions to check on their understanding of the mathematics if these points have not already been brought out:

Q What functions are of special significance in the study of exponential growth? Describe their key features.

Q How is x found in equations of the form $a^x = b$?

Summary for students

- Exponential models are important applications of mathematics; many situations exhibit exponential growth or decay, or have logarithmic scales of measurement.
- The three basic functions (and their graphs) $y = e^x$, $y = e^{-x}$ and $y = \ln x$ are key to understanding exponential models.
- $y = e^x$ and $y = \ln x$ are each inverse functions of the other. Values of these functions can be found using a calculator.
- Exponential growth curves grow very rapidly; exponential decay curves decrease very rapidly.
- The natural logarithm $y = \ln x$ curve is negative for $0 < x < 1$ and grows very rapidly until $x = 1$, where it is zero; it flattens off as x tends to larger values.
- The solution of the equation $a^x = b$ is $x = \ln b / \ln a$.

12.2 Foundation

Transforming rectilinear figures

Objectives

- Transform rectilinear figures using:
  - translations;
  - reflection about a line;
  - rotation about a centre of rotation;
  - enlargement about a centre of enlargement.
- Understand the meanings of positive, negative and fractional scale factors in enlargements.
- Use ICT to investigate the generation of geometric patterns, including Islamic patterns.

Starter

Vocabulary

rectilinear
planar
Cartesian grid
axis of symmetry
transformation
reflection
translation
rotation
rigid

Resources

Mini-whiteboards

Tell the class that the lesson is about transforming planar, rectilinear shapes.

Q What is the meaning of *rectilinear*? Give an example of a rectilinear shape? Give an example of a non-rectilinear shape.

Q What is the meaning of *planar*? Write down an example of a planar shape. Write down an example of a non-planar shape.

Q What is an *axis of symmetry* of a planar shape? Give an example of a planar shape with at least one axis of symmetry.

Explain to the class that some geometrical objects are rigid and that such shapes cannot change their size or shape even when moved about.

Q What happens to a rigid planar shape when it is moved anywhere in its own plane?

Q Some geometrical objects can change their shape. Can the class suggest ways in which a planar geometric object might not be rigid? (the shape could be made of rubber, for example)

Q What can be done to a non-rigid planar object to change its shape or size? (it can be stretched or distorted in some other way)

Tell students to try to visualise what you tell them.

They are to imagine that they have a rigid square shape, ABCD, of side 2 units. The shape is placed on a Cartesian coordinate grid that is calibrated in the same units. Originally, vertex A is at the point (4, 3). The shape is now moved one unit to the right and three units up in its own plane. Tell them that such a movement is an example of a *translation*.

Q What are the new coordinates of the vertex A?

Q What, if anything, can be said about the new coordinates of the other three vertices B, C and D? (the vertices B, C and D could be anywhere, depending how the square is originally oriented about the vertex A)

Q What other translations could be applied to vertex A?

Q Does it make any difference to the final position of the square if the same translation is applied at different points in the square?

Q What translation will send the square with its vertex A at the point (4, 3) to a position where A is at the point (–1, –7)?

Now tell the class that the square is placed with vertex A at the point (0, 0) and vertex B at the point (2, 0).

Q What can be said about the coordinates of the vertices C and D? (not enough information to decide between the points (0, 2) and (2, 2) or the points (0, –2) and (2, –2))

Tell them the square is now positioned with one vertex at the origin and two of its adjacent sides along the positive x-axis and along the positive y-axis respectively. The square is then reflected about the line $x = 2$.

Q What are the new coordinates of all four vertices of the square?

Q What is the relation of the square in its new position compared to the square in its original position?

Tell the class that the next visualisation is more complicated.

Tell them that a thin, straight stick of length 15 cm lies in a plane and one end is loosely attached to a point A. The stick can rotate in the plane about this point.

Q What path does the other end of the stick trace out as the stick rotates about A? (circle, centre at A, radius 15 cm)

Tell the class that A is at the origin of a coordinate grid measured in centimetre units and that the free end of the stick is at the point (–12, 9). The stick is then rotated 90° in a clockwise direction.

Q What are the new coordinates of the free end of the stick? (9, 12)

Now ask the class to imagine that the end of the stick not attached at A is attached rigidly to a rigid triangle.

Q What happens to this triangle as the stick rotates about A? (the triangle rotates rigidly with stick)

Q What happens to each vertex of the rigid triangle as the stick rotates about A? (each vertex describes a circle about A)

Q Does the shape of the triangle change as the stick rotates about A? (no)

Q Does the orientation of the triangle relative to the stick change as the stick rotates about A? (no)

Finally, ask the class to imagine that each vertex of the triangle is attached to the point A by a tight, straight thread and that the stick rotates clockwise through an angle of 90°.

Q What angle does each thread rotate about A? (90° clockwise)

Q Does the angle between the stick and any one thread change as the stick and the triangle rotate? Explain your answer. (no, because the whole set up rotates together)

Q What happens to the stick and the threads if the stick now rotates through 45° in an anticlockwise direction? Through an arbitrary angle θ in a clockwise direction? (each thread rotates clockwise through an angle of θ)

Q Do the threads remain in the same orientation with respect to the stick? Give a reason for your answer. (yes, because the system is rigid and all parts rotate together)

Q Why is it important to specify the direction of a rotation? (depending on the sense of the rotation, in general the image will end up in different positions for a given size angle)

Main activity

Vocabulary

enlargement
scale factor
negative scale factor
image
isometry

Resources

Resource 12.2a, one per student
OHTs 12.2b to 12.2d

Explain to the class that the lesson is about moving planar rectilinear shapes in their own plane. Use the word *translation* to describe a movement in which the whole shape slides along a straight line from one position to another. Tell the students that, when conventional axes are used, a translation represented by the notation $\begin{pmatrix}-2\\-3\end{pmatrix}$ means 'translate 2 units to the left and 3 units down', whereas $\begin{pmatrix}4\\2\end{pmatrix}$ means 'translate 4 units to the right and 2 units up'. Get the students to suggest the meaning of a translation in which one of the entries is positive and one is negative, and another in which at least one of the entries is zero.

Give students a copy of diagram 1 on **Resource 12.2a**. Ask them to perform a few simple translations on the shape, e.g. $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$, drawing the image of the original shape after each translation. Ask students to suggest a meaning for the word *image*.

Q What has stayed the same and what has changed when these translations have been applied to the original shape?

Explain that the image is identical in shape and size to the original and is therefore congruent to the original shape. A translation preserves lengths and angles and is an example of an *isometry*, a word coined from two ancient Greek words *isos* meaning 'equal' and *metron* meaning 'measure'.

Show diagram 2 on **OHT 12.2b** and ask students to explain carefully what they see. Make sure that the idea of a transformation of the original shape by a series of reflections about specific lines in the plane is established.

Q What reflections are needed to successively transform the original shape into the additional three images in the slide?

Now show diagram 3 on **OHT 12.2b** and invite students to comment on the diagram. Steer discussion towards the idea of a clockwise rotation about the origin of the coordinate grid; the diagram shows successive clockwise rotations of 90° about the origin. You could remind students of the mental visualisation at the start of the lesson.

Q What constructions can be performed to find the image of each vertex under this rotation? (draw lines to the origin from each vertex on the original double triangle figure and rotate each line clockwise by 90°)

Explain to students that a rotation in the plane is an example of what is called a *linear transformation*, often simply called a *transformation*, and that all rotations are specified by three pieces of information: the angle and direction of rotation, and the centre of rotation, which is the point about which the rotation takes place.

Now show diagram 4 on **OHT 12.2c** and inform students that the figures in the slide can be produced in several ways from the original figure. Ask:

Q How can the figures in diagram 4 be produced by a series of reflections and translations on the original figure?

Q How can the figures in diagram 4 be produced by a series of reflections and rotations of the original figure?

Remind the class that two transformations have been introduced so far: reflection about a line and rotation about a centre of rotation. Now introduce a third transformation: an enlargement about a centre of enlargement. Show diagram 5 on **OHT 12.2d** and ask students to explain what they see. Bring the discussion together pointing out that of the three additional figures in the diagram, compared to the original double triangle, one is enlarged by a *scale factor* of 2, and one has been reduced to half size (scale factor $^1/_2$), with the smallest figure half the size of the latter, and that in all cases the origin is the centre of enlargement. Another way to look at the diagram is that, starting from the smallest figure, a sequence of images has been generated each of which is an enlargement of the previous image, with scale factor 2 and centre of enlargement at the origin.

Q What would happen if the sequence of images on OHT 12.2d were extended through the origin into the fourth quadrant?

Now show diagram 6 on **OHT 12.2d**.

Q What is going on in this diagram?

Make sure that the ensuing discussion highlights two features of the diagram, namely that the first image is a rotation of the original figure by 180° about the origin and that the second figure is an enlargement of the first image by a scale factor of 2 and with centre of enlargement at the origin. Explain that a rotation through 180° followed by an enlargement with scale factor k is called enlargement by the negative scale factor $-k$.

Q What is the scale factor of the larger of the two images in the diagram?

Further tasks

Resources

Dynamic geometry software

OHTs 12.2e and 12.2f

Many extensions can be developed from the basic activity of this lesson.

Diagram 7 on **OHT 12.2e** could be used to develop the idea of reflection in lines more complicated than lines simply parallel to either of the grid axes.

Diagram 8 on **OHT 12.2e** could be used to illustrate what happens when parts of the original figure in diagram 1 (resource 12.2a) can be dragged. To do this effectively some dynamic geometry software will be needed. Dragging can be used to introduce yet another type of transformation known as a *shear*.

Using a dynamic geometry software package, patterns of considerable complexity may be generated from simple planar shapes by using combinations of all the various transformations and translations.

Complicated patterns may be analysed to see how they have been generated from simple basic shapes. For example, **OHT 12.2f** illustrates an Islamic pattern found in a mosque in Isfahan in Iran. Students should be encouraged to analyse patterns and to generate complex patterns themselves.

Some more able students might be set a research project to investigate the interesting patterns of the twentieth-century Dutch artist M. C. Escher.

Consolidation

Ask the class to recall and describe the different transformations discussed in the lesson.

Ask which movements and transformations leave the image congruent to the original figure.

Point out that in a reflection about a line, the line acts as an axis of symmetry for the original shape taken together with its image under the reflection.

Ask which transformations create an image which is similar but not congruent to the original figure.

Ask students to recall the meaning of key words used in the lesson.

Summary for students

- A *translation* slides an object in a plane from one position to another. Every translation can be described by a vector that denotes the movements in the x- and y-directions.
- Three common transformations are *reflection*, *rotation* and *enlargement*. Every rotation is specified by a centre and an angle of rotation. Every enlargement is specified by a scale factor and a centre of enlargement. Every reflection is about a specified line.
- Combinations of transformations and enlargements done in a particular order are needed to move from a basic figure to some more complicated image of that figure.

12.3
Quantitative methods

Using the binomial distribution

Objectives

- Recognise when to use the binomial distribution and know how to identify the probability of success, p, and the probability of failure, $(1 - p)$; know the notation $X \sim B(n, p)$ for a random variable X modelled by the binomial distribution.
- Know that the sum of all probabilities in a binomial distribution totals to 1. Calculate binomial probabilities and expected frequencies for different numbers of successes.
- Calculate the mean and variance of the binomial distribution as $\mu = np$ and $\sigma^2 = np(1 - p)$; use the mean and variance to model sample data expected to have a binomial distribution.
- Understand the principle of a hypothesis test involving a null hypothesis or alternative hypothesis, and use the related vocabulary of significance level, one-tail or two-tail test, critical value, critical region, acceptance region.
- Set up and perform a hypothesis test on a binomial probability distribution model, identifying the null hypothesis and the alternative hypothesis, and make correct inferences from the test.

Starter

Vocabulary
inference
parameter
statistic
hypothesis test
null hypothesis
alternative hypothesis
significance level
one-tail test
two-tail test
critical value
critical region
acceptance region

Start the lesson with review questions on the principles and vocabulary of hypothesis testing.

Q What is meant by *statistical inference*? How is this different from the meaning of *inference* in everyday language?

Q What is the parameter of the binomial distribution? How does this relate to the mean and the variance of the distribution?

Q What is the key difference between a parameter and a statistic?

Q What notation is used for the null and alternative hypotheses? What are their purposes? How are they formulated?

Q How can you tell from the alternative hypothesis whether the test is one- or two-tailed?

Q How are the critical value, the critical region and the acceptance region related?

Discuss the significance level in some detail, emphasising that it is the probability below which an event is considered too rare to have happened by chance under the null hypothesis H_0. However, with a 5% significance level and a correct null hypothesis, if we repeated the hypothesis test 20 times, on average one time in those 20 we would make the wrong decision by rejecting H_0. So the conclusion of a hypothesis test is never certain – that is why the significance level is always stated as part of the conclusion. Students tend to react to this by suggesting that it is therefore a good idea to reduce the significance level. This is a good point at which to introduce the idea that reducing the significance level increases the probability of making a different error – accepting the null hypothesis when in fact the alternative hypothesis is true. So there is a trade off (see Further tasks).

Ask students to discuss in pairs what differences they already understand between the use of the word *significance* in statistics, compared with its use in everyday language. Take feedback. Emphasise that *significant* in the statistical sense does not mean 'important' – it simply means 'not likely to happen by chance'. Also introduce the idea that the common choice of 0.05 for the significance level is in some senses arbitrary and so, within practical research, a result that only 'tends towards significance' may be worth pursuing further.

Main activity

Resources

OHT 12.3a

Resources 12.3b, 12.3c

Tell the class that this lesson will enable them to consolidate their understanding of the principles and vocabulary of hypothesis testing, and enable them to set up and perform hypothesis tests on binomial probability distribution models in practical situations.

Show **OHT 12.3a**. Ask students to discuss the questions in pairs and between them construct clearly reasoned written responses. Share the responses among the group, for example by students writing their responses on to OHTs or large sheets of paper. As a class, discuss the strengths and weaknesses of the responses, in particular teasing out any misconceptions. Agree a model answer.

Resources 12.3b and **12.3c** give instructions for carrying out two practical experiments leading to hypothesis tests on binomial probability distribution models. You may wish to choose one practical for the whole class to do in groups or have different groups doing the two practicals. The titles of the practicals are 'Are you telepathic?' and 'Taste test'. Check that students set up the practicals correctly and make sure their system for recording the results is accurate.

Give the students time to attempt to set up and perform the hypothesis test. Note the following key values:

Are you telepathic?

$H_0 : p = {}^1/_4$; $H_1 : p < {}^1/_4$; one-tailed test; $P(X \geq 7) = 0.014\,34$; $P(X \geq 6) = 0.054\,44$, where X is the number of correctly identified suits from the 12 cards

Strictly, a 5% critical region would not contain the value 6, but it seems sensible in this case to go over the 5% and include 6 in the critical region. This is a useful discussion point.

Taste test

$H_0 : p = {}^1/_3$; $H_1 : p > {}^1/_3$; one-tailed test; one tailed test; $X \sim B(n, {}^1/_3)$ under H_0

Ask a group of students to present their hypothesis test to the rest of the class and invite comments from other groups.

Further tasks

Students comfortable with the principle and vocabulary of hypothesis testing could be asked to research the concepts of type I and type II errors, and the power of a test.

Consolidation

Ask students to reflect on how their understanding of the principles and vocabulary of hypothesis testing has developed during the lesson.

Q What is the purpose of performing a hypothesis test?

Q How could you explain the meaning of a significance level to someone who knows no statistics?

Q Is it ‘better’ or ‘more interesting’ to find that you reject the null hypothesis?

Q Someone claims that you can never know anything for certain in science. To what extent do you agree? How does your knowledge of hypothesis testing help you understand the point that was being made?

Summary for students

- Hypothesis tests are part of statistical inference. Statistical inference enables conclusions to be drawn from sample data about the population from which the sample is taken, and probability to be used to say how confident we are that our conclusions are correct.
- Hypothesis tests assess evidence from the sample data about the unknown parameters of distributions.
- There is a standard way to set up and perform a hypothesis test.
- The conclusion to a hypothesis test should relate back to the original problem, and refer to the significance level.

12.4 Mathematics for science

Applications of calculus (revision)

Objectives

- Use the derivative to explore a range of optimisation problems in which a function is maximised or minimised.
- Understand integration as the inverse process to differentiation.
- Understand that the area bounded by a positive function $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$, with $a \le x \le b$, is the definite integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.
- Use the terminology that if z is a function of x then the derivative of z with respect to x is $\frac{\mathrm{d}z}{\mathrm{d}x}$ and the differential of z is the symbolic expression $\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}x}\mathrm{d}x$; understand that z and its differential can be used to replace the variable of integration in an integral.
- Solve a range of physical problems involving simple differential equations for exponential growth and decay.

Starter

Vocabulary

derivative
definite Integral
indefinite integral
differential
composite function
fundamental theorem of calculus

Resources

Resource 12.4a

Start the lesson with some quick recall and interpretation questions (reproduced on **Resource 12.4a**).

Q What is the derivative of e^{kx}?

Q What is the derivative of $\ln x$?

Q What is $\int \sec^2 x\,\mathrm{d}x$?

Q What is $\int \frac{1}{x-a}\mathrm{d}x$?

Q How can $\int \frac{1}{x^2-a^2}\mathrm{d}x$ be calculated?

Q What is the derivative of the product uv, where u and v are both functions of x?

Q A composite function is defined by $\mathrm{h}(x) = \mathrm{g}(\mathrm{f}(x))$, where g and f are each functions of x. What is the derivative of $\mathrm{h}(x)$?

Q $y\frac{\mathrm{d}y}{\mathrm{d}x}$ is the derivative with respect to x of what function of x?

Q What distinguishes a definite integral from an indefinite integral?

Q If z is a function of x, what is a representation for the differential of z?

Q What is the interpretation of the derivative at a point on the graph of a function?

Q How is a derivative calculated from first principles?

Q What is the fundamental theorem of calculus?

Q What is the essential characteristic of a point of inflexion?

Q What conditions usually distinguish a local maximum of a function from a local minimum?

Main activity

Resources

OHTs 12.4b, 12.4c, 12.4e

Resources 12.4d and 12.4f, one of each per student

Tell the class that this consolidation lesson will review a range of question types that cover aspects of both differential and integral calculus, including the solution of differential equations.

Begin with a discussion of areas under curves.

Q How are areas under curves calculated?

Q Two curves intersect at two points. How is the area between the curves calculated?

Show **OHT 12.4b**. Explain that the graph shows three curves C_1, C_2 and C_3 and that their equations have not been specified. Tell the class that one of the curves represents the integral function of another of the curves.

Q Is it possible to decide which are the two curves in question? How?

Tell the class to assume that curves C_1 and C_3 are both quadratic functions.

Q What are the equations of the three curves?

Show the key steps in the working, using appropriate notation.

Ask students to discuss in pairs how they would calculate the area of the region bounded by each of the curves. Ask one pair to explain their method. Ask if any other pair has a different method, and if so ask them to explain their method. Repeat as necessary.

Now ask them to calculate the area (the answer is $^1/_2$).

Now show **OHT 12.4c** (do not show the diagram at this stage), and read aloud the question posed on the OHT. Ask students what they need to know to begin to solve the problem. Write the sensible suggestions on the board and get students to criticise the less sensible suggestions. From the list of suggestions ask students for any relevant formulae that might be required and get them to suggest ways of getting into the problem with some relevant mathematics. A key to progress is to ensure that the importance of drawing an appropriate diagram is recognised, because this simple re-representation of the question will provide the relevant clue for establishing the correct mathematical model of the situation. The bottom part of **OHT 12.4c** can be used to demonstrate an appropriate drawing to represent the cross-section of the cone within the sphere.

Give the students time to attempt the problem and arrive at a full solution. They could work in small groups and then one group can be asked to present its solution to the rest of the class and invite comments on their solution. After a sufficient while present them with a photocopy of the solution (**Resource 12.4d**).

Q Why, when setting the derivative to zero, cannot $\sin\theta=0$ or $\cos\theta=-1$?

Q Is it clear that $\cos\theta = {}^1/_3$ gives the maximum value for V_c?

Stress the importance of concentrating on the essentials of the problem and of compact working; stress also the use of the implication sign to move from one step to the next.

Continue with **OHT 12.4e** as an example of a type of differential equation that has to be solved in real-world problems about radioactivity. Question students to probe their understanding of the equation.

Q Do you recognise this equation? What is it?

Q What method can be used to solve this differential equation?

Ask students to solve the equation and be prepared to discuss their solutions.

When the solution has been demonstrated to most students' satisfaction invite them to consider the questions:

Q How is the half-life calculated?

Q What units will the half-life be measured in?

Give the students a few minutes to calculate an expression for the half-life and hence determine the value of the decay constant and the dimensions of its measurement. **Resource 12.4f** can given to students as a model solution.

Further tasks

Resources

Resources 12.4g and 12.4h

Students who need either to work faster or slower should be set easier or harder problems to work on in their own time, either in class or as homework.

Use **Resource 12.1g** for some harder examples and **Resource 12.4h** for some easier examples on similar themes to those explored within this lesson.

Consolidation

Ask the students what they have learned in the lesson.

Get the students to pose questions that will help them get into the problems they are attempting to solve. Steer the discussion if necessary. These are the sorts of questions they should be asking:

- What is needed to solve this problem?
- What mathematics is relevant to the solution?
- What formulae will be needed?
- Are there different ways of thinking about the problem?
- Might a diagram be helpful?

Now ask students:

Q If the problem is about optimising a function, what will be two key steps in the working?

Q If the problem is about the properties of curves, what mathematics will be helpful in the analysis?

Q If the problem involves differential equations, what process is needed to solve these equations?

Q What differential equation is used to model radioactive decay?

Q What other situations might be modelled using similar mathematics?

Q If the problem is about finding areas under curves, what is the relevant mathematics?

Summary for students

- Derivatives are used to maximise or minimise functions.
- Differentiation is used to explore properties of functions.
- Integration is used to undo differentiation, and to find areas under curves.
- Integration is also used to solve differential equations.
- Differential equations of the form $\frac{dy}{dx} = ky$, where k is any non-zero constant, give rise to exponential solutions and are common in many real-world applications.

