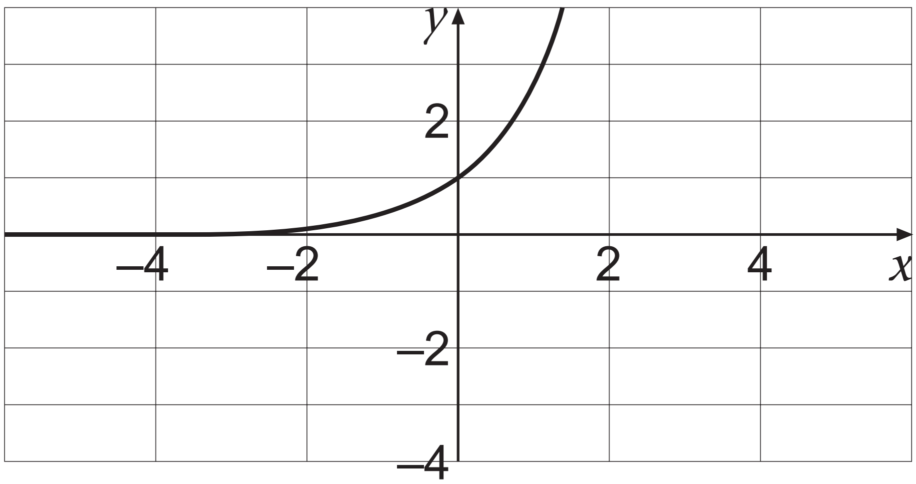
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–4
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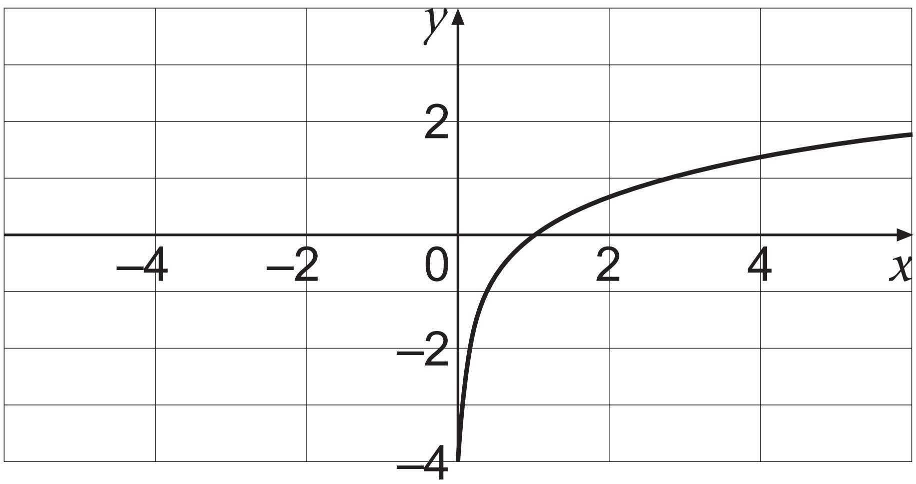
y
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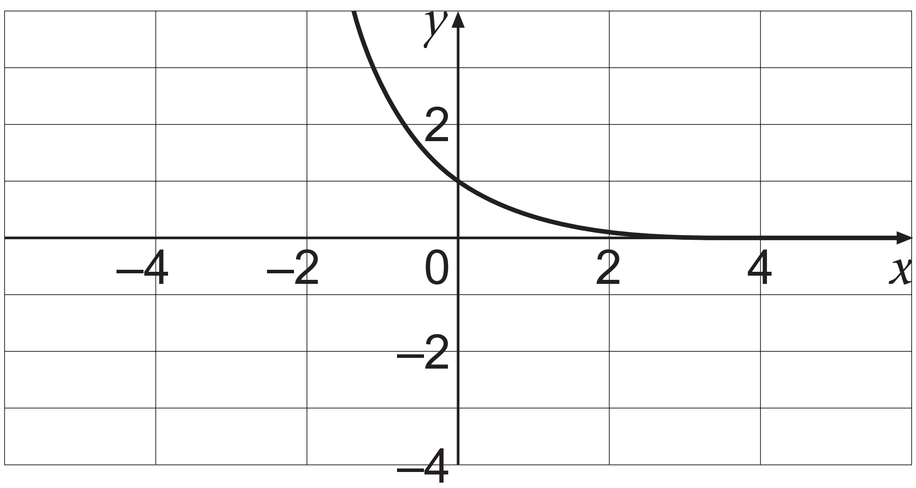

y
2
–4
–2
0
2
4
x
–2
–4

The population of the United States of America from 1790 to 2000

Year	Population	Year	Population
1790	3 929 214	1900	75 994 575
1800	5 308 483	1910	91 972 266
1810	7 239 881	1920	105 713 620
1820	9 638 453	1930	122 775 046
1830	12 866 020	1940	131 669 275
1840	17 069 453	1950	150 697 361
1850	23 191 876	1960	179 323 175
1860	31 443 321	1970	203 302 031
1870	39 818 449	1980	226 545 805
1880	50 155 783	1990	248 709 873
1890	62 947 714	2000	281 421 906

Statistics from US Census Bureau

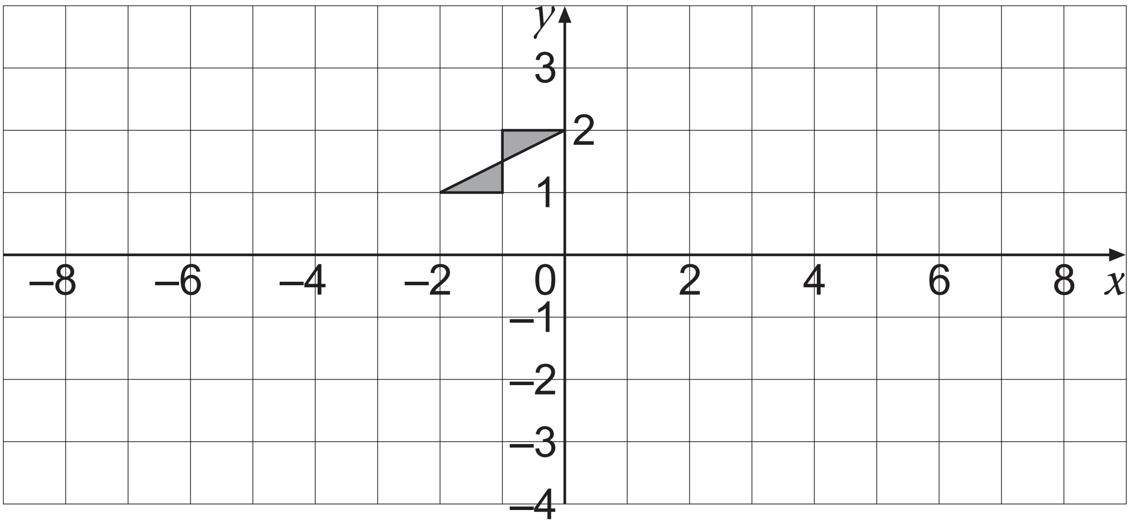


Diagram 1

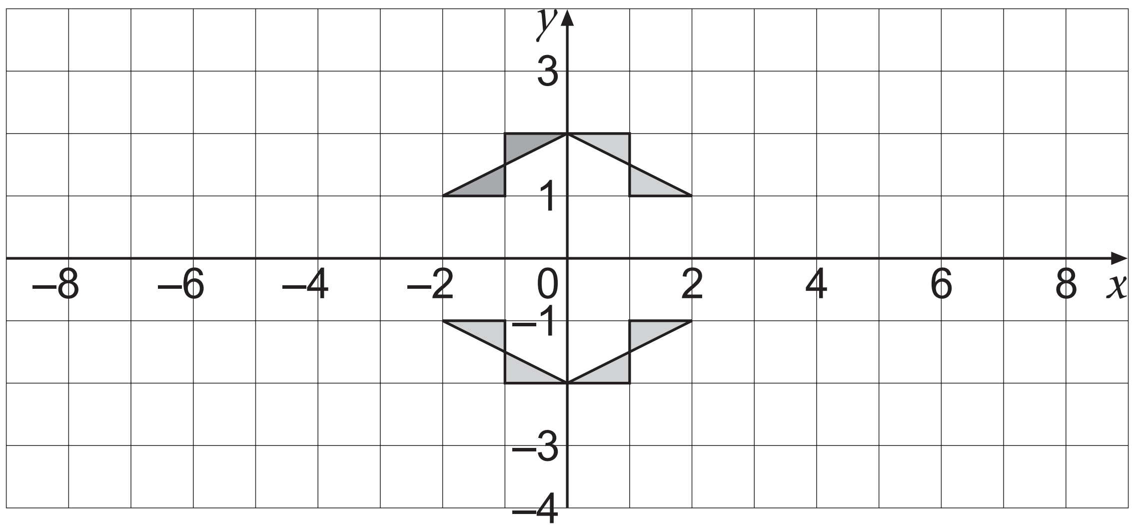


Diagram 2

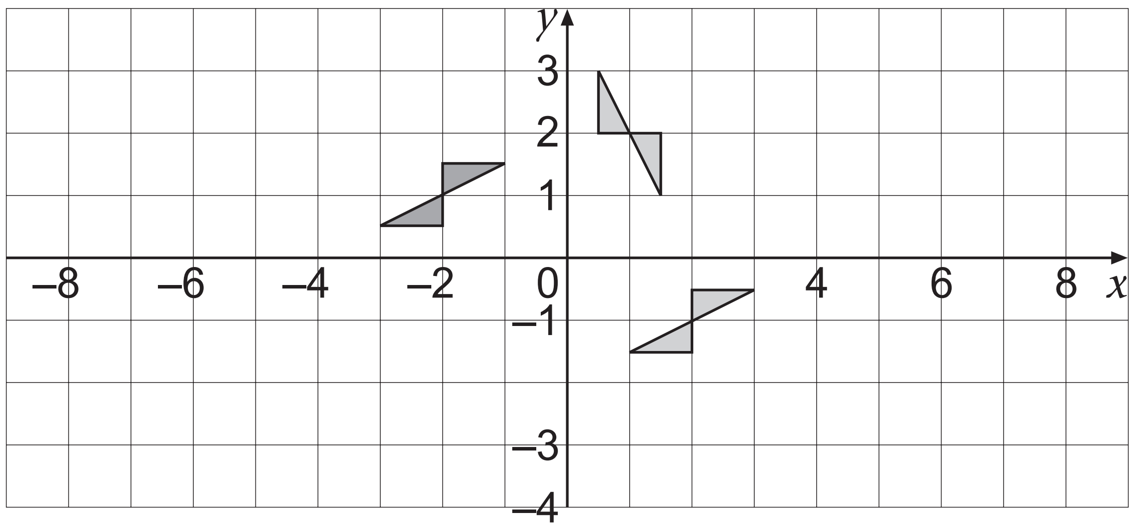
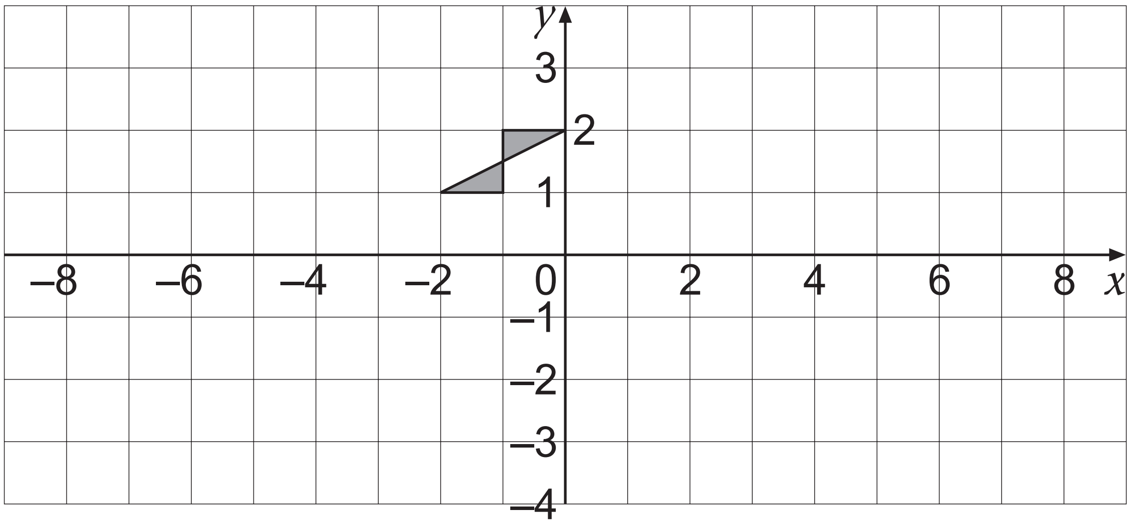


Diagram 3



Original figure

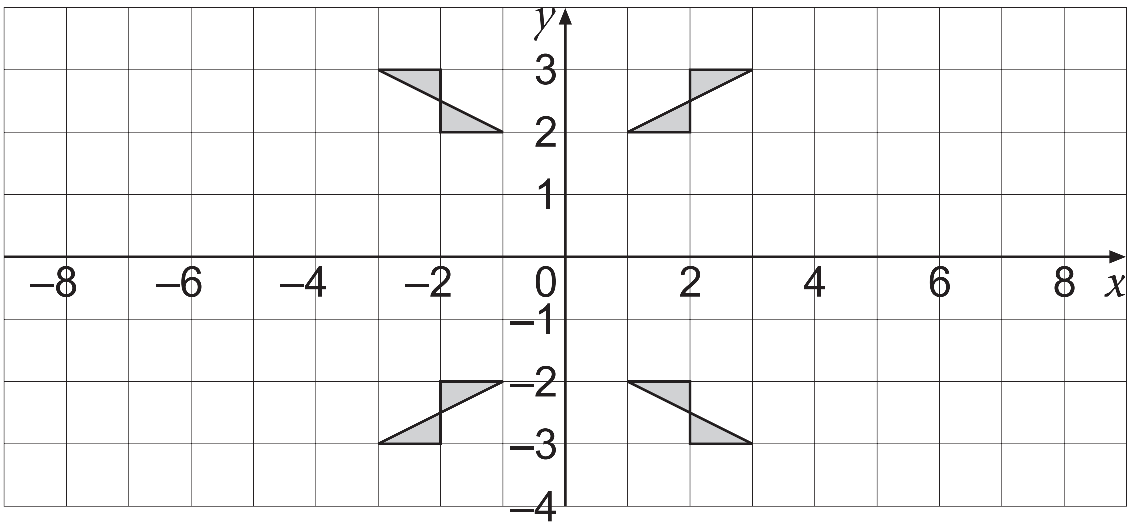


Diagram 4

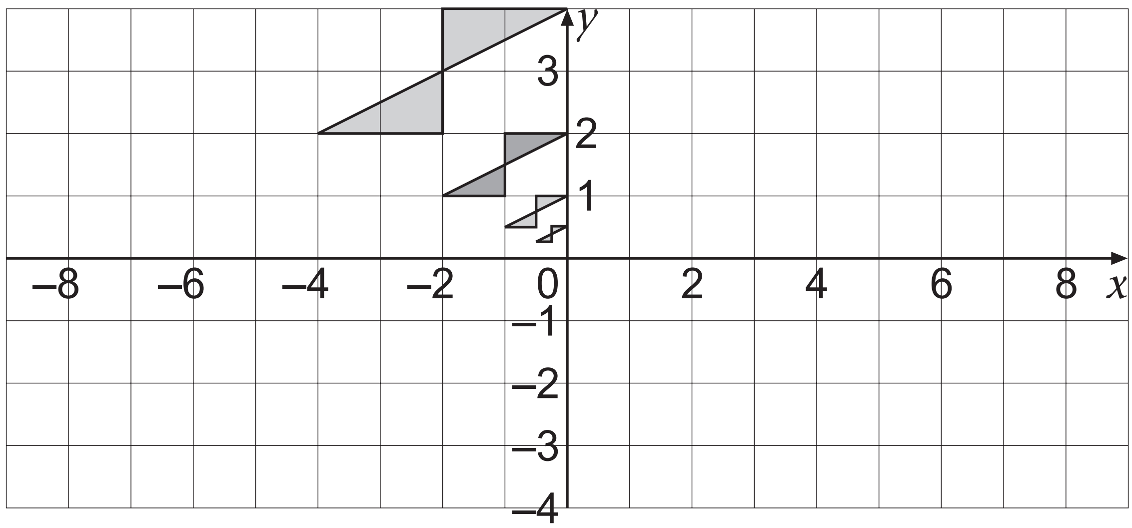


Diagram 5

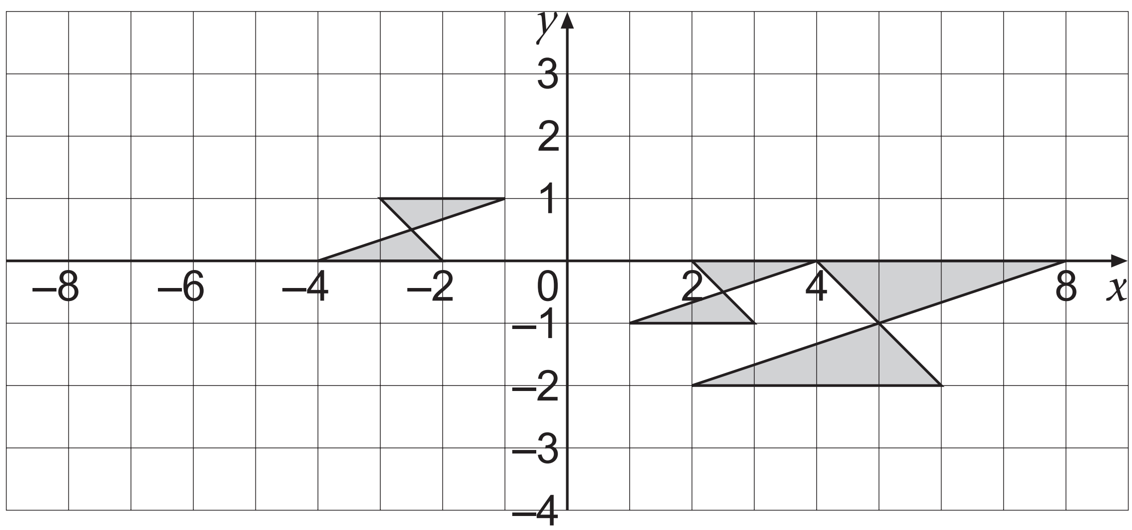


Diagram 6

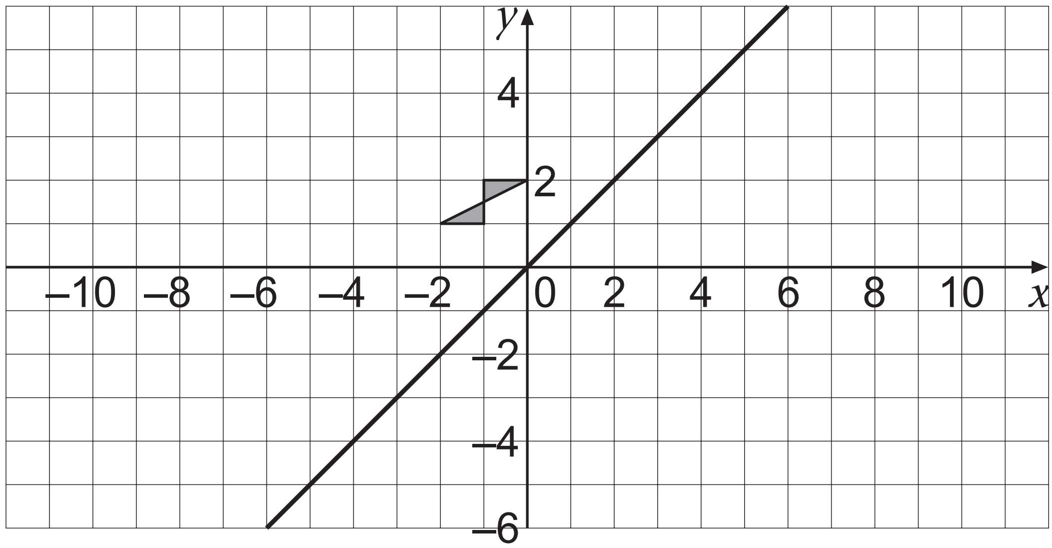


Diagram 7

j

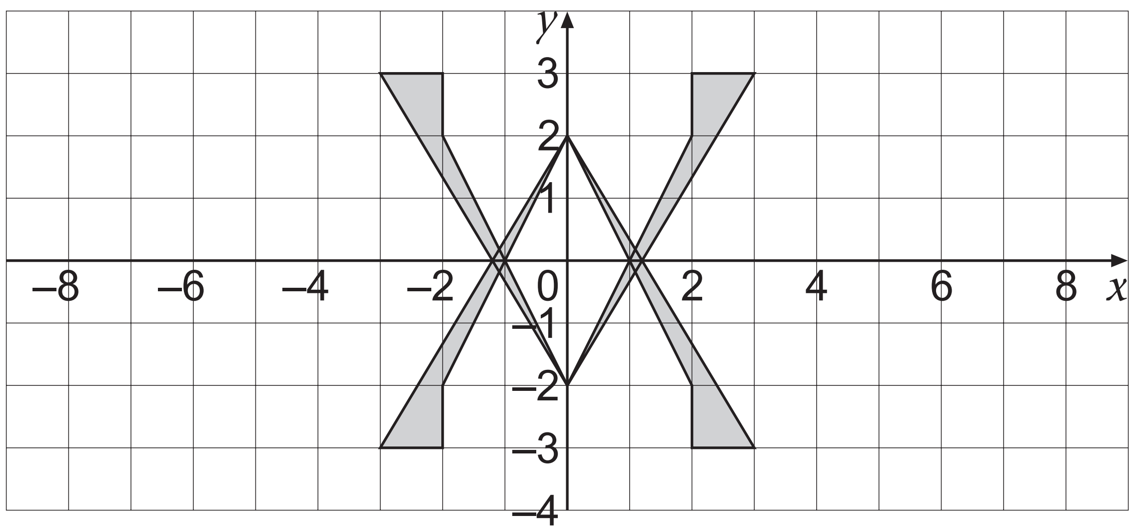
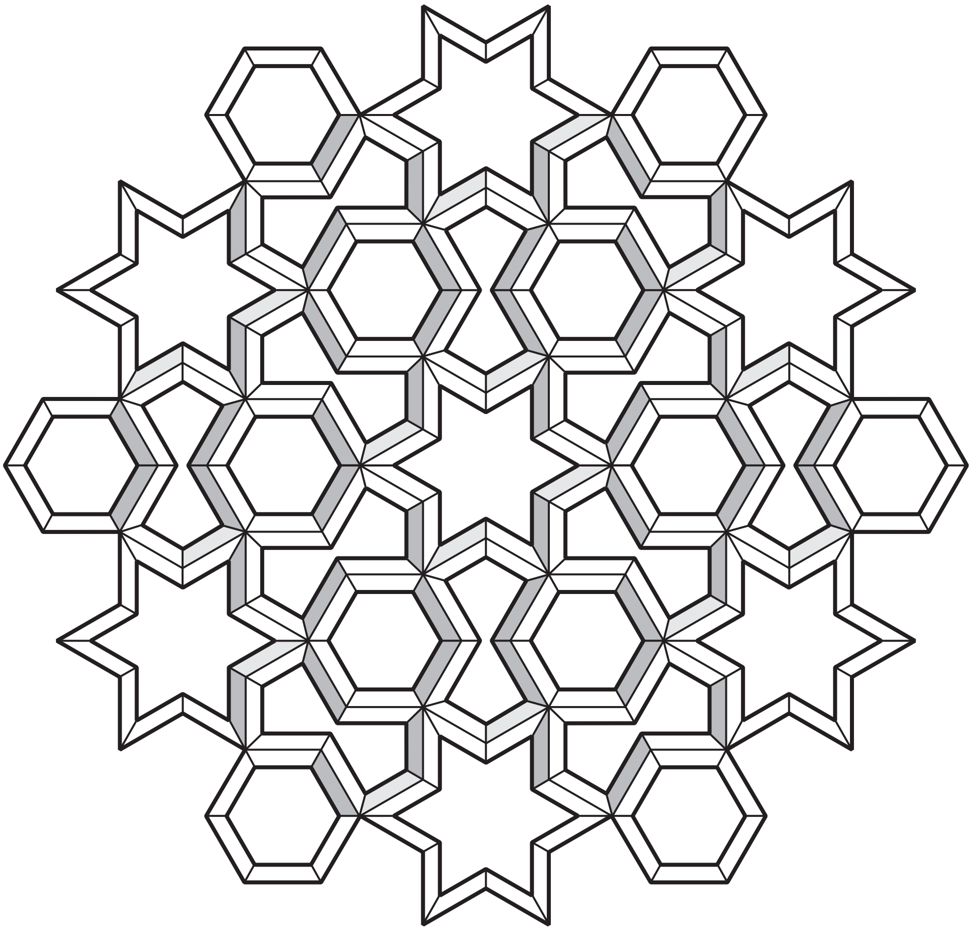


Diagram 8



Pattern from a mosque in Isfahan

When asked to explain the meaning of 'statistical significance at the 0.05 level', a student says:

'This means there is only a probability of 0.05 that the null hypothesis is true.'

Is this an essentially correct explanation?
Explain your answer carefully.

Another student is asked why statistical significance appears so often in research reports. The student says:

'Because saying that results are significant tells us that they cannot easily be explained by chance alone.'

Is this an essentially correct explanation?
Explain your answer carefully.

Are you telepathic?

You need to work in a group of 3: organiser, subject 1 and subject 2. You also need: a stopwatch, a pack of playing cards and a way of generating random numbers.

Before the experiment, the organiser uses random numbers to select 12 cards and the order they will be used in the experiment. It is only the suit that matters so random numbers 1 and 2 could be club, 3 and 4 spades etc. The organiser then gives the pile of 12 cards to subject 1 face down.

The organiser says ‘First card’ and starts timing 30 seconds. During this time, subject 1 turns over the first card and concentrates hard on it. Subject 2 must not be able to see the card. After 30 seconds, subject 2 must write down the suit of the card.

Repeat with the remaining cards.

At the end, count up the number of correct responses by subject 2.

Now set up a hypothesis test to determine whether subject 2 is telepathic. Remember to state your null and alternative hypotheses. Is this a one- or two- tailed test? What is the critical region? Using your results, what do you conclude?

Taste test

You need two people to organise the trials and record the results and between 10 and 20 subjects to perform the trials.

The taste test can be between any two types of similar products; for example, two brands of cola drink, two types of sparkling mineral water, two brands of chocolate. The samples of the products should look as alike as possible.

For each trial, present the subject with three samples, two from one product and one from the other. The position of the odd sample should be randomised to prevent the order of tasting having an effect and a coin should be thrown to decide which product is presented twice. The subject is asked to pick out the odd sample. The subject may have a sip of water between each tasting. Ask the subject if he/she can detect a difference and, if so, which is the odd sample. Record the response.

Repeat the trial with between 10 and 20 subjects. Ignore all the subjects who could not detect a difference. Count the number n who could detect a difference and the number x who correctly identified the odd sample.

Now set up a hypothesis test to determine whether people can taste the difference. What do you conclude?

1 What is the derivative of e^{kx}?

2 What is the derivative of $\ln x$?

3 What is $\int \sec^2 x \, dx$?

4 What is $\int \frac{1}{x-a} dx$?

5 How can $\int \frac{1}{x^2 - a^2} dx$ be calculated?

6 What is the derivative of the product uv, where u and v are both functions of x?

7 A composite function is defined by $h(x) = g(f(x))$, where g and f are each functions of x. What is the derivative of $h(x)$?

8 $y \frac{dy}{dx}$ is the derivative with respect to x of what function of x?

9 What distinguishes a definite integral from an indefinite integral?

10 If z is a function of x, what is a representation for the differential of z?

11 What is the interpretation of the derivative at a point on the graph of a function?

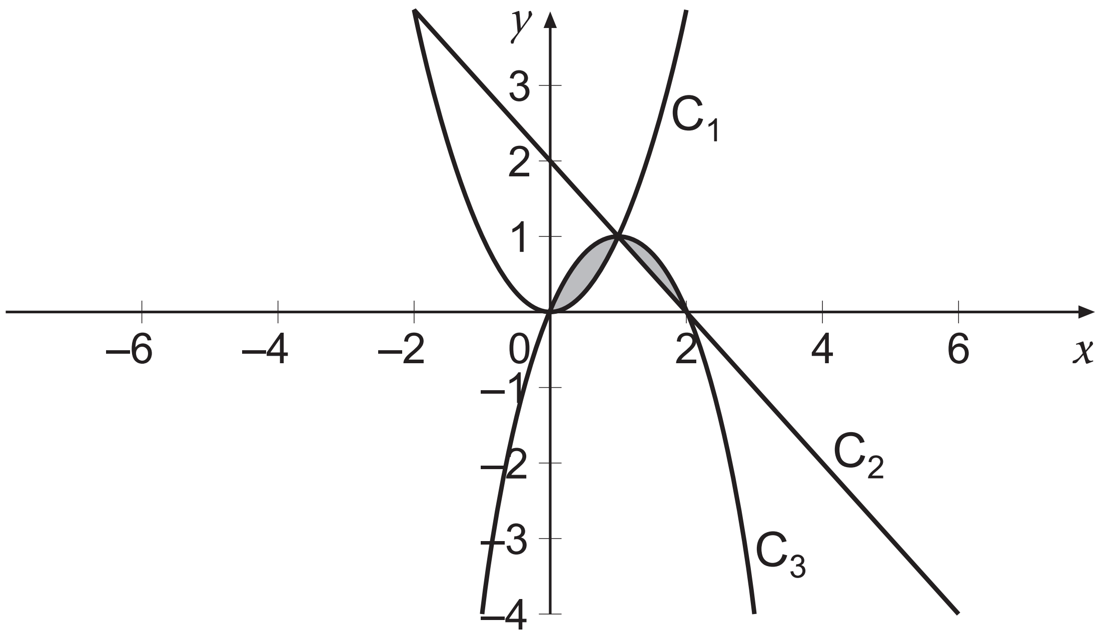
12 How is a derivative calculated from first principles?

13 What is the fundamental theorem of calculus?

14 What is the essential characteristic of a point of inflexion?

15 What conditions usually distinguish a local maximum of a function from a local minimum?

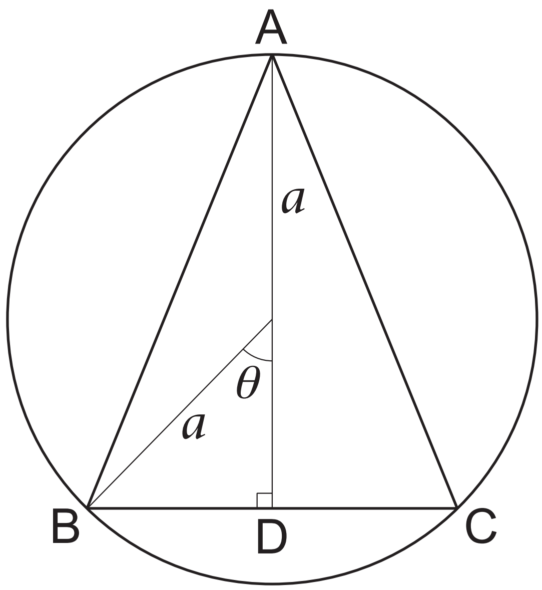
OHT 12.4b



Intersecting curves

A right circular cone lies inside a hollow sphere with its vertex and the edge of its base touching the inside of the sphere. The sphere has radius *a* cm.

Find the condition for the volume of the cone to be greatest and calculate the fraction of the volume of the sphere that is occupied by the cone of greatest volume.



Cross-section of cone sitting in sphere

Formulae:

Volume of cone $V_c = \frac{1}{3}\pi a^2 h$

Volume of sphere $V_s = \frac{4}{3}\pi a^3$

Identity $\sin^2\theta \equiv 1 - \cos^2\theta$

Now

$$h = a(1+\cos\theta)$$

$$\Rightarrow \quad V_c = \tfrac{1}{3}\pi a^3 \sin^2\theta(1+\cos\theta)$$

$$\Rightarrow \quad 3V_c = \pi a^3(1-\cos\theta)(1+\cos\theta)^2$$

$$\Rightarrow \quad 3\frac{dV_c}{d\theta} = \pi a^3[\sin\theta(1+\cos\theta)^2 - 2\sin\theta(1-\cos\theta)(1+\cos\theta)]$$

$$= 3\pi a^3 \sin\theta(1+\cos\theta)^2(3\cos\theta - 1)$$

To maximise V_c, set the derivative to zero:

$$\frac{dV_c}{d\theta} = 0 \Rightarrow \cos\theta = \tfrac{1}{3}$$

Hence the maximum volume is found from
$3V_{max} = \frac{32}{27}A$, where $A = \pi a^3$.

So $\dfrac{V_{max}}{V_s} = \dfrac{\frac{32}{72}}{4} = \dfrac{8}{27}$.

The number of radioactive emitters in a radioactive element is modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN$$

where t is the time in years and k is a positive constant called the *decay constant*.

A measure of the longevity of a radioactive element is its *half-life*, which is the time it takes for the number of radioactive emitters to decrease by one half starting at any time in the element's evolution. It does not matter when a half-life is measured from – it is always the same.

Radium is a radioactive element. Its half-life is about 1550 years.

Solve the differential equation.

Find the decay constant of radium and state the units the decay constant is measured in.

Let N_0 be the number of radioactive emitters when t = 0, and c a constant. Then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN$$

$$\Rightarrow \quad \int \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}\mathrm{d}t = -k\int \mathrm{d}t$$

$$\Rightarrow \quad \int \frac{1}{N}\mathrm{d}N = -kt + c$$

$$\Rightarrow \quad \ln N = -kt + c$$

$$\Rightarrow \quad N = \mathrm{e}^{-kt+c} = \mathrm{e}^{-kt}\mathrm{e}^{c} = N_0\mathrm{e}^{-kt}$$

Let T be the half-life in years, measured from time zero. Then at time T the number of radioactive emitters is $N_0/2$. So

$$\frac{N_0}{2} = N_0\mathrm{e}^{-kT}$$

$$\Rightarrow \quad \mathrm{e}^{kT} = 2$$

$$\Rightarrow \quad kT = \ln 2$$

$$k = \frac{\ln 2}{T}$$

The dimensions of the decay constant are $(\text{years})^{-1}$.

The value of k can be approximated as 0.000 447 19 by substituting
T =1550 into the expression for k.

Further questions

(Questions 5 to 8 are easier than questions 1 to 4.)

1 The population of a country has been modelled by the differential equation $\frac{dy}{dt} = ky(P - y)$, where k is constant, y is the population at time t years and P is the maximum population the country can support in terms of food and other resources. Show that if the population is $P/2$ when t = 0 then the solution of the differential equation is $y = \frac{P}{1 + e^{-Pkt}}$. Sketch the graph of this function. After how many years will the population reach 0.75P?

2 Use integration of the function $y = x^k$ and the concept of area under a curve to establish the result

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + ... + n^k}{n^{k+1}} = \frac{1}{k+1}$$

3 Two corridors meet at right angles. One corridor is of width x metres and the other has width y metres. A large board of length l metres has to be moved around the corridors with its upper edge kept horizontal. Calculate the maximum length of board that can make the turn from one corridor to the other, giving the answer in terms of x and y.
The corridors have vertical walls of height z metres. What is the longest pole that can be moved from corridor to corridor?

4 The keel of a boat is the frame which defines its cross sectional shape. The keel of a yacht has the geometric shape bounded by the curves $\int_0^1 \frac{1}{\sqrt{1 + x^2}}\,dx$, $y = \frac{1}{9}x^2$, y = 1 and $x = 2y(1 - y)$. Sketch the cross-sectional shape of the yacht and determine the area of this cross-section, Assume all lengths are measured in metres.

5 A rectangle has one side of length $2x$ units. The rectangle is placed inside a semicircle with radius 10 units in such a way that one side, of length $2x$, lies along the diameter of the semicircle and the vertices of the parallel side each just touch the inside of the circle. What are the dimensions of the largest rectangle that satisfies these conditions?

6 Calculate the volume of revolution when the area bounded by the curve $y = \frac{1}{x-1}$ and the lines $x = 3$ and $x = 4$ is rotated about the x-axis.

7 A parachutist falls to Earth with vertical velocity v (in metres per second) given by the formula $v = \frac{mg}{k}(1 - e^{-kt/m})$, where m is the mass of the parachutist in kilograms, g is the acceleration due to gravity (measured in m s^{-2}) and k is a constant. The parachutist jumps out of the aeroplane at height H above the ground. Find the parachutist's height above the Earth's surface as function of the time t.

8 Evaluate the following integrals:

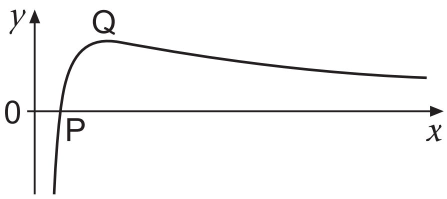
a. $\int_2^3 \frac{4x}{(x^2-1)}\,dx$

b. $\int_0^{\pi/3} \sin x \cos^4 x\,dx$

c. $\int_0^1 \frac{e^x}{1+e^x}\,dx$

d. Use the substitution $x = \tan\theta$ to determine $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$.

The figure shows a sketch of the graph $y = f(x)$, where $f(x) = \dfrac{\ln x}{x}$ $(x > 0)$.



The graph crosses the x-axis at the point P and has a turning point at Q.

Write down the x-coordinate of P.

Find the first and second derivatives $f'(x)$ and $f''(x)$, simplifying the answers as far as possible.

Hence show that the x-coordinate of Q is e.

Find the y-coordinate of Q in terms of e.

Find $f''(e)$ and use this to show that Q is a maximum point.

Find the exact area of the finite region between the graph of $y = f(x)$, the x-axis and the line $x = 2$.

MEI Specimen Paper 2004